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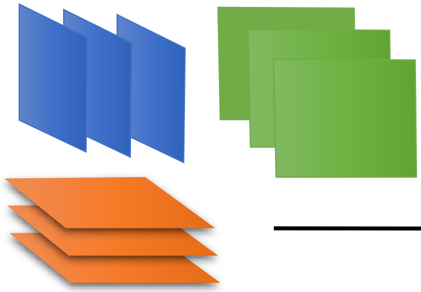

# Tensor Modeling Based Wireless Communications


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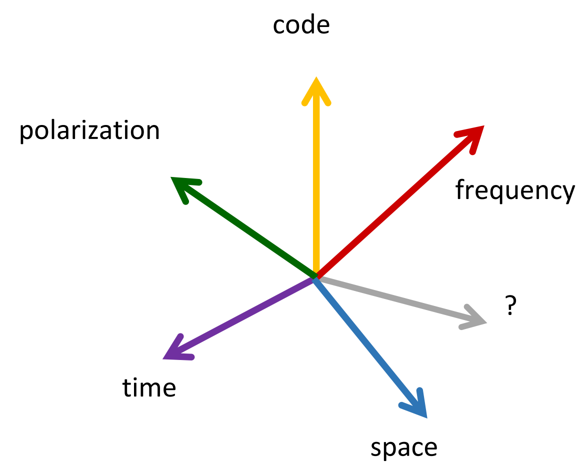
November 23, 2022

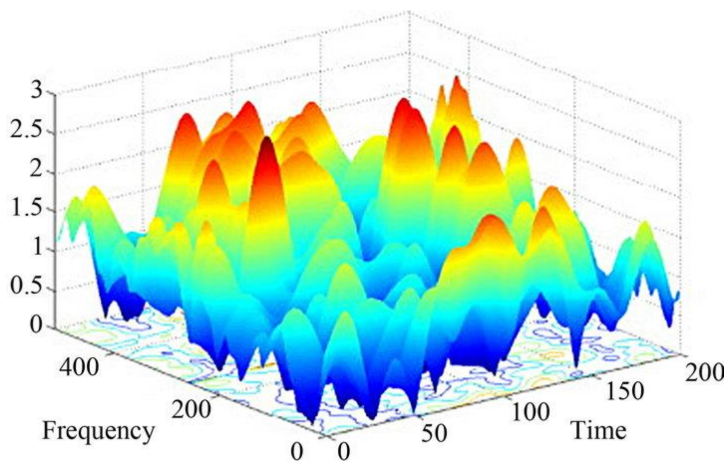


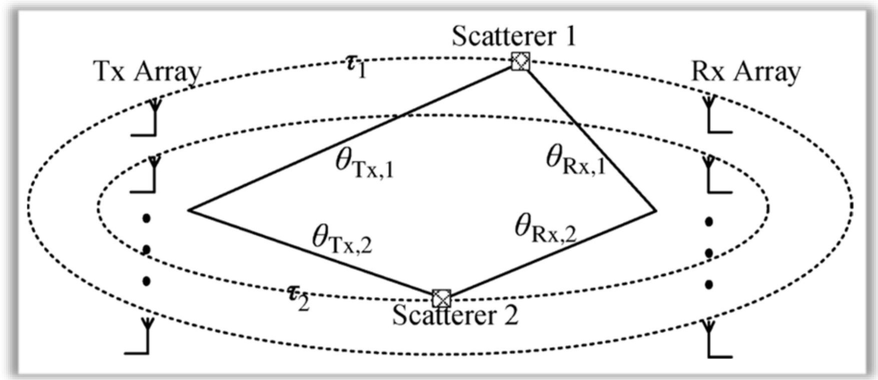
# Disclaimer

A bit of everything

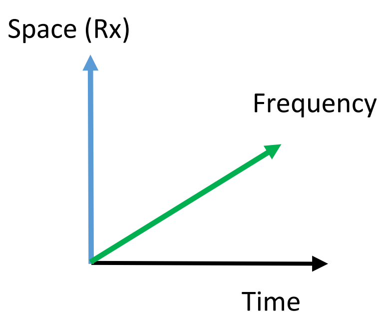
# Intro

**The wireless channel has a multidimensional nature!**

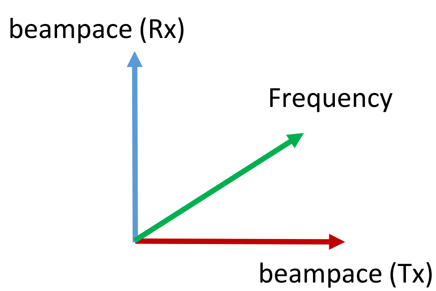


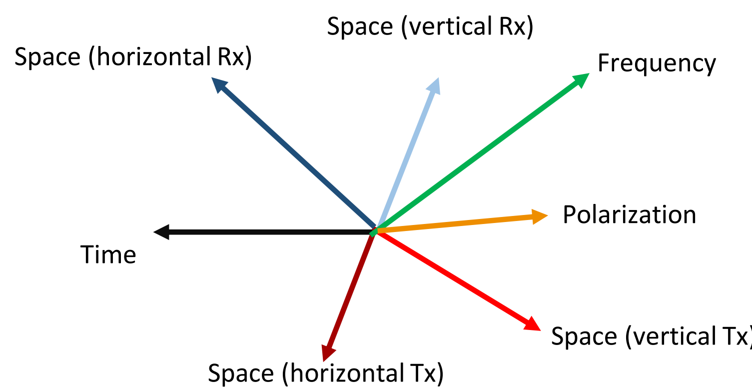

**3D channel**



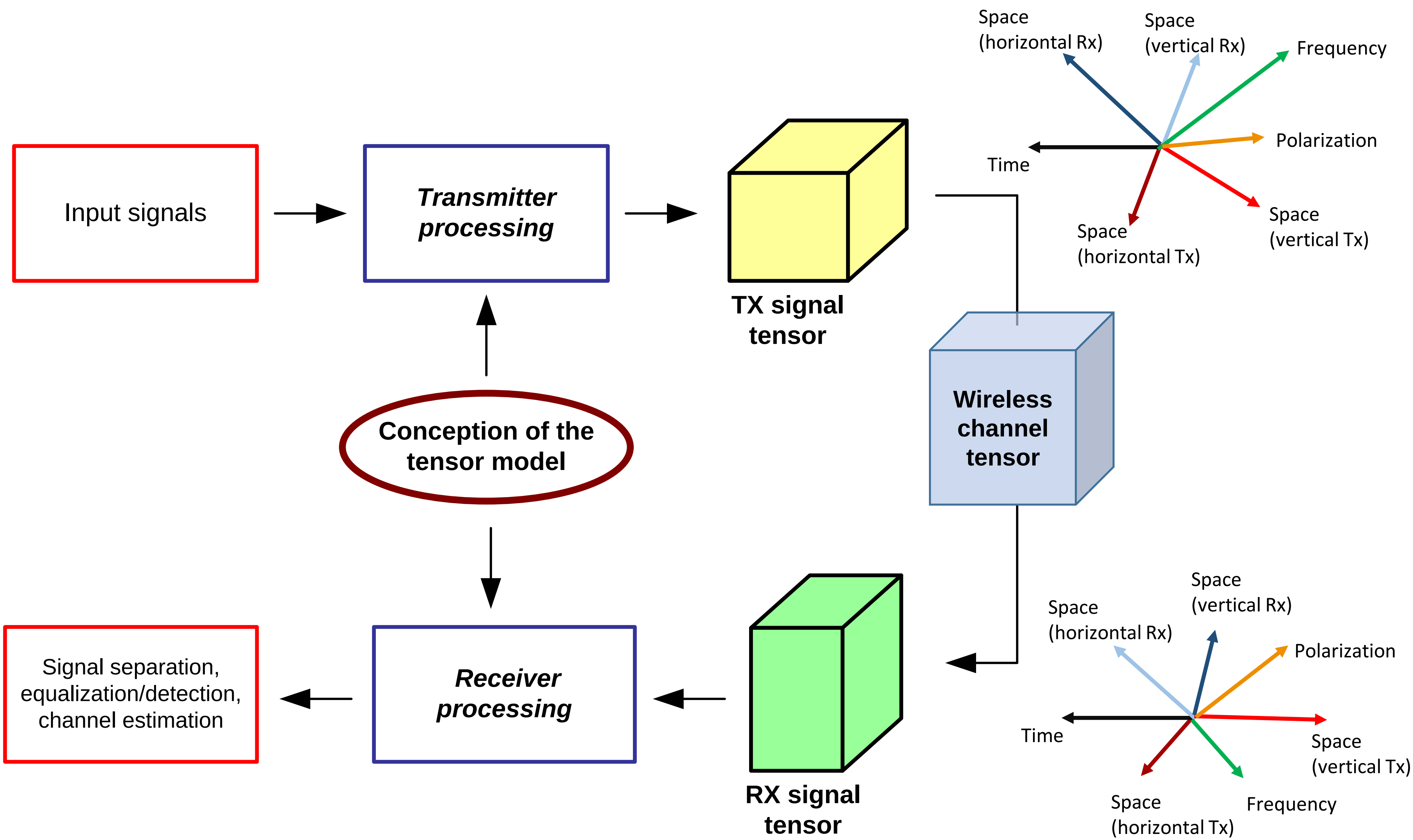

**3D channel**




**7D channel**



# Tensor perspective to wireless communications



# Tensor signal processing in wireless comm.

**Key features**

- Exploit the multidimensional structure of the channel and its multiple forms of diversity
- Blind/semi-blind channel estimation & symbol detection under more relaxed conditions (compared to matrix-based SP)
- Complexity management of large-scale filter optimizations (e.g. massive MIMO, beamforming, equalizers..)
- Noise-relisient multidimensional constellation designs

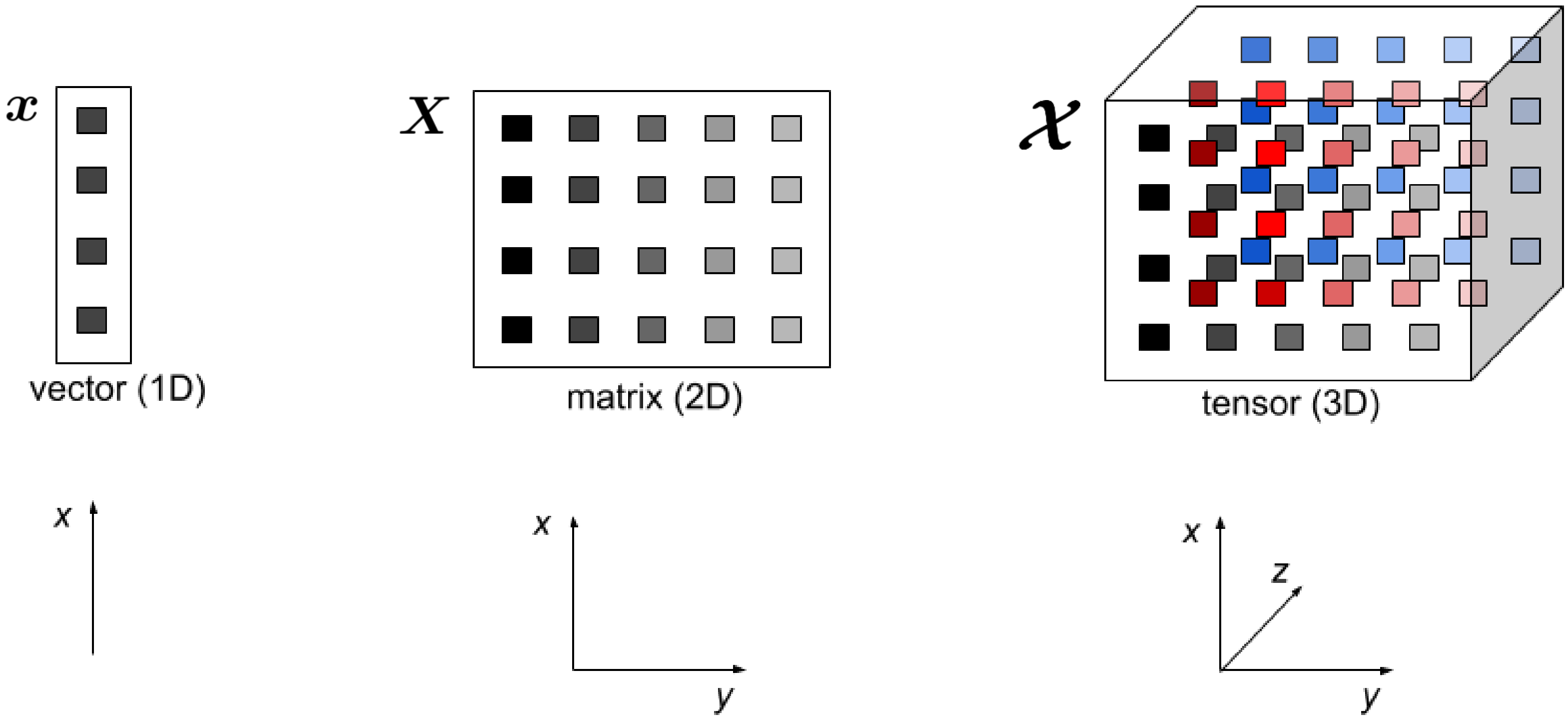
# Outline

- Channel modeling & estimation
- Space-time-frequency MIMO schemes
- Relay-based communications
- Multi-linear beamforming design
- Multi-linear constellation design

# A bit of tensor decompositions

# What is a Tensor?

- An intuitive definition…



# What is a Tensor?

- A "nicer" mathematical definition 🙂

$$\boldsymbol{\mathcal{X}} = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} x_{i,j,k} \underbrace{(\boldsymbol{e}_i^{(I)} \circ \boldsymbol{e}_j^{(J)} \circ \boldsymbol{e}_k^{(K)})}_{(i,j,k)\text{-th coordinate}}$$

$$\boldsymbol{e}_i^{(I)} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{-th position}$$

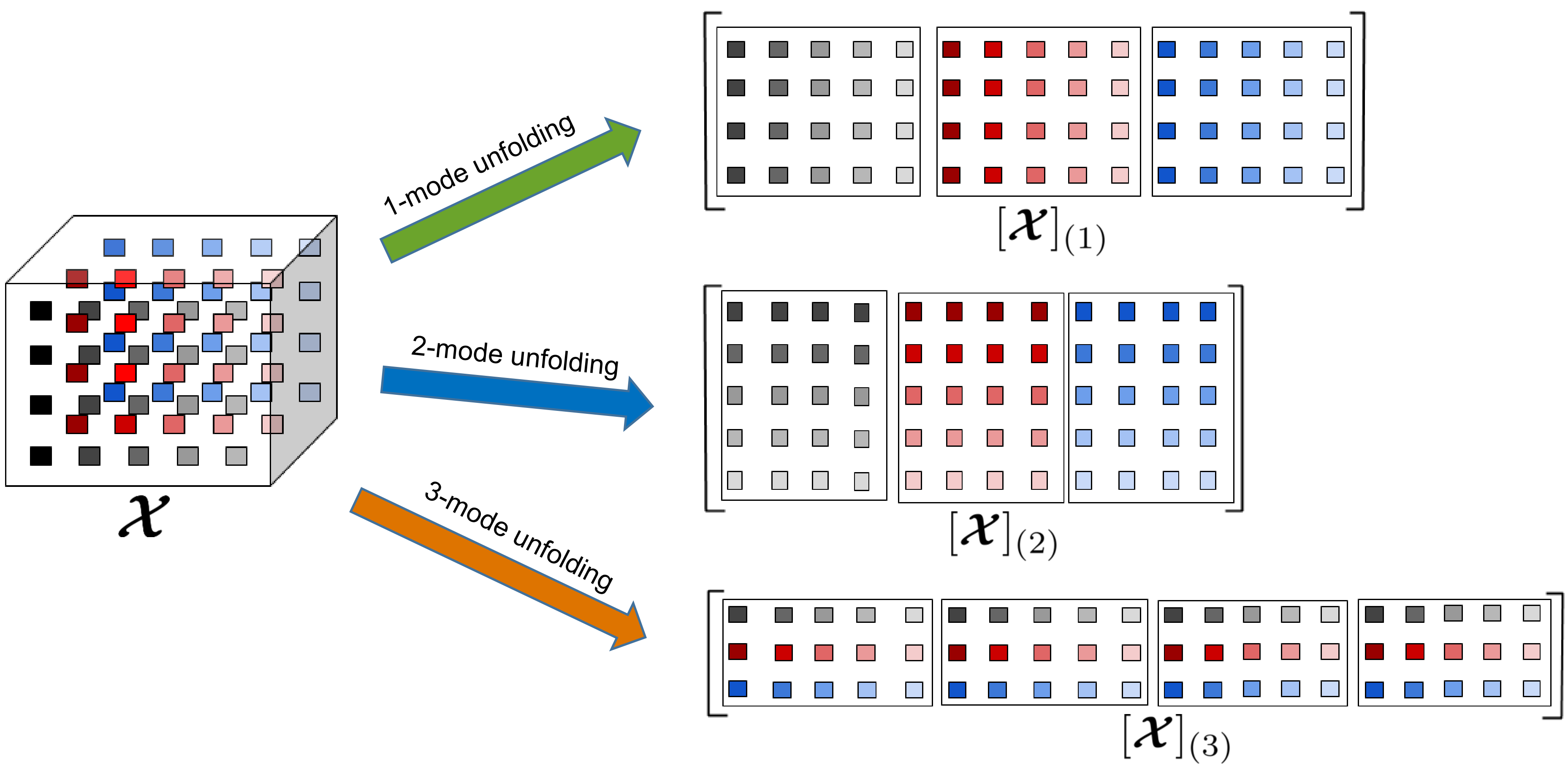
$\circ$ : outer product

- Tensor as a multi-linear map

$$T(\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}) = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} x_{i,j,k}(\boldsymbol{u}_i \circ \boldsymbol{v}_j \circ \boldsymbol{w}_k)$$

$$\boldsymbol{U} = [\boldsymbol{u}_i]$$
$$\boldsymbol{V} = [\boldsymbol{v}_j]$$
$$\boldsymbol{W} = [\boldsymbol{w}_k]$$

# Unfolding a tensor into matrices



# An useful operator: The *n*-mode product

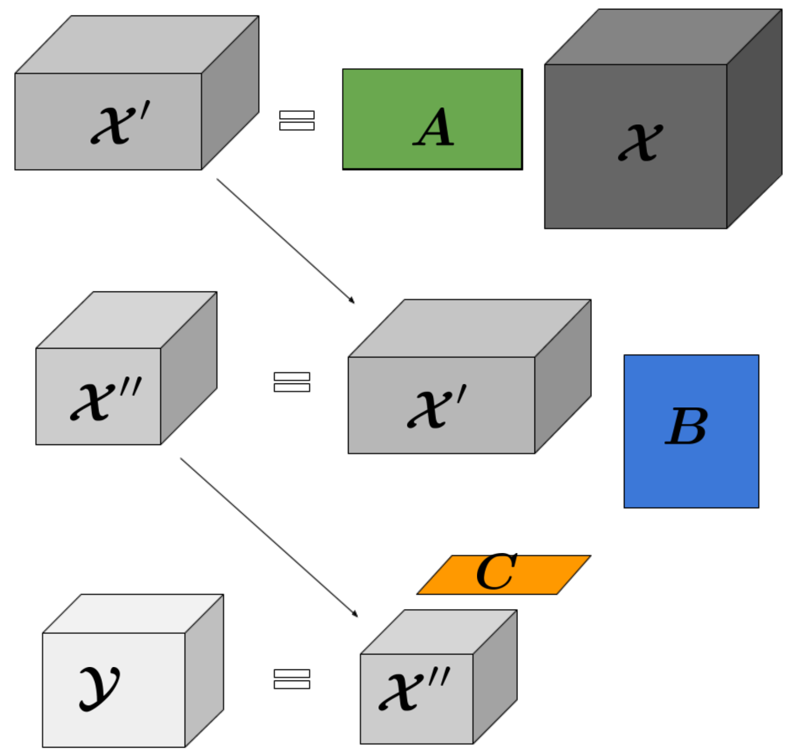
- Defines a product between a tensor and a matrix (or vector)

$$\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{X}} \times_n \boldsymbol{A} \quad \leftrightarrow \quad [\boldsymbol{\mathcal{Y}}]_{(n)} = \boldsymbol{A}[\boldsymbol{\mathcal{X}}]_{(n)}, \qquad \forall n$$

[De Lathauwer et al. '2000]
[Kolda & Bader, 2009]

- Multiple *n*-mode products

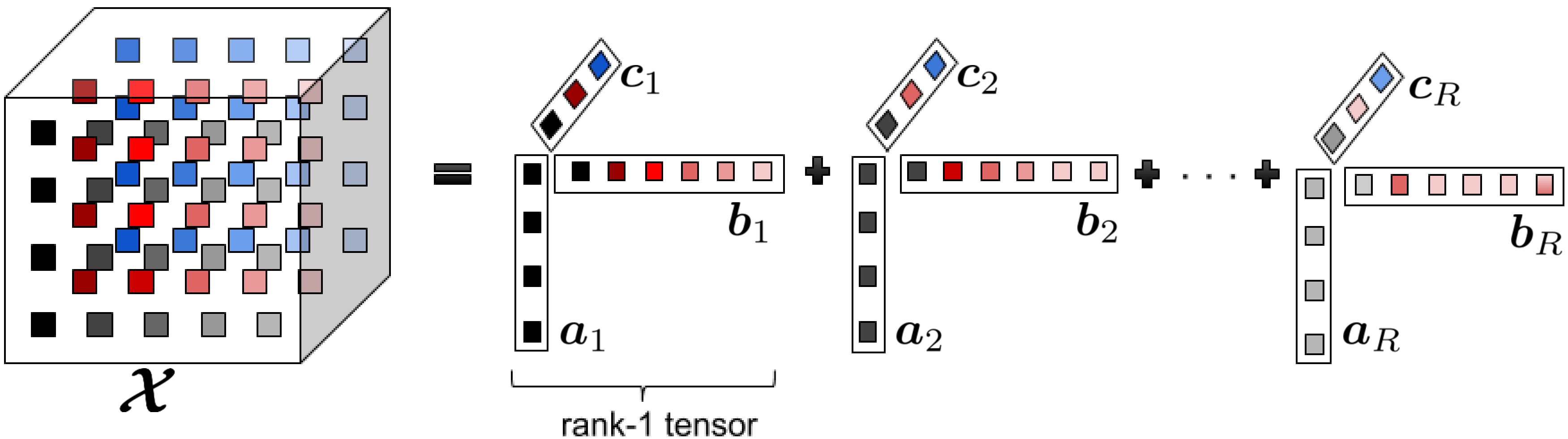
$$\boldsymbol{\mathcal{Y}} = (\underbrace{(\underbrace{(\underbrace{\boldsymbol{\mathcal{X}} \times_1 \boldsymbol{A})}_{\boldsymbol{\mathcal{X}}'} \times_2 \boldsymbol{B})}_{\boldsymbol{\mathcal{X}}''} \times_3 \boldsymbol{C})}_{\boldsymbol{\mathcal{Y}}}$$




*Concept of "multi-linear compression"*

# The “canonical” tensor decomposition

- Decomposition in a sum of rank-1 components




Also known as:

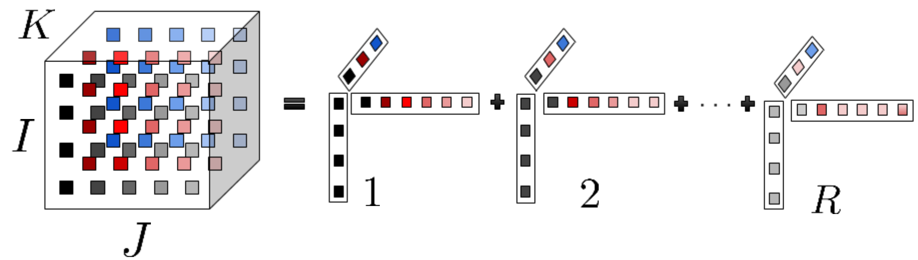
- Canonical polyadic decomposition (CPD) [Hithcock'1927]
- Parallel Factor decomposition (PARAFAC) [Harshman'1970] [Carroll & Chang'1970]

Tensor rank $R$ → minimum # of rank-1 tensors yielding $\mathcal{X}$ in a combination

# Canonical polyadic decomposition (CPD)

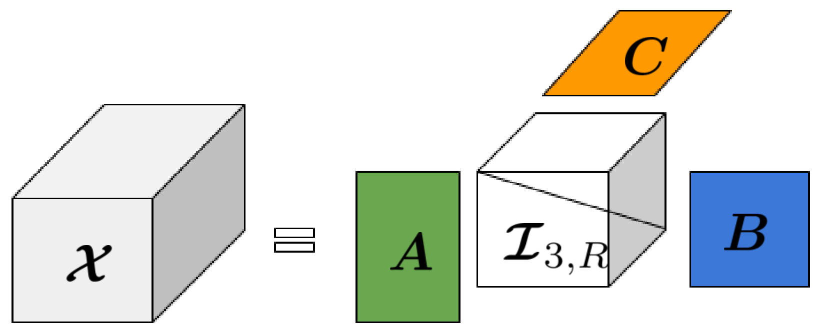
- Outer-product notation

$$\boldsymbol{\mathcal{X}} = \sum_{r=1}^{R} \boldsymbol{a}_r \circ \boldsymbol{b}_r \circ \boldsymbol{c}_r$$




- *n*-mode product notation

$$\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{I}}_{3,R} \times_1 \boldsymbol{A} \times_2 \boldsymbol{B} \times_3 \boldsymbol{C}$$




- "Vectorized" form

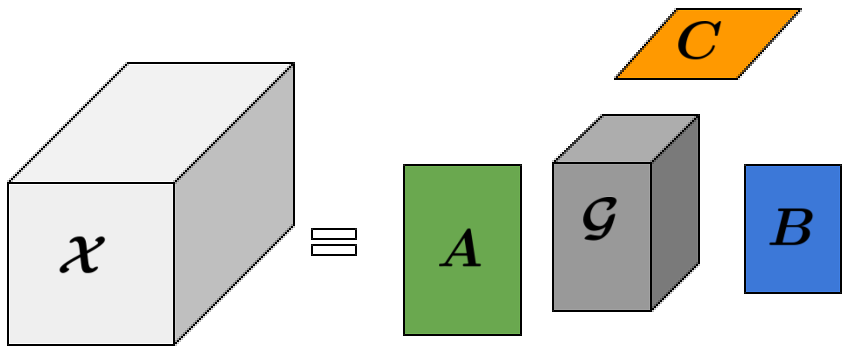
$$\boldsymbol{x} = (\boldsymbol{A} \diamond \boldsymbol{B} \diamond \boldsymbol{C})\boldsymbol{1}_R$$

$\diamond$ : Khatri-Rao product

$$\boldsymbol{A} = [\boldsymbol{a}_r]$$
$$\boldsymbol{B} = [\boldsymbol{b}_r]$$
$$\boldsymbol{C} = [\boldsymbol{c}_r]$$

# Tucker decomposition

Full multi-linear map $\boldsymbol{\mathcal{X}} = \sum_{p=1}^{P} \sum_{q=1}^{Q} \sum_{r=1}^{R} g_{p,q,r} (\boldsymbol{a}_p \circ \boldsymbol{b}_p \circ \boldsymbol{q}_r)$ [Tucker'1966]




$$\boldsymbol{A} = [\boldsymbol{a}_p]$$
$$\boldsymbol{B} = [\boldsymbol{b}_q]$$
$$\boldsymbol{C} = [\boldsymbol{c}_r]$$

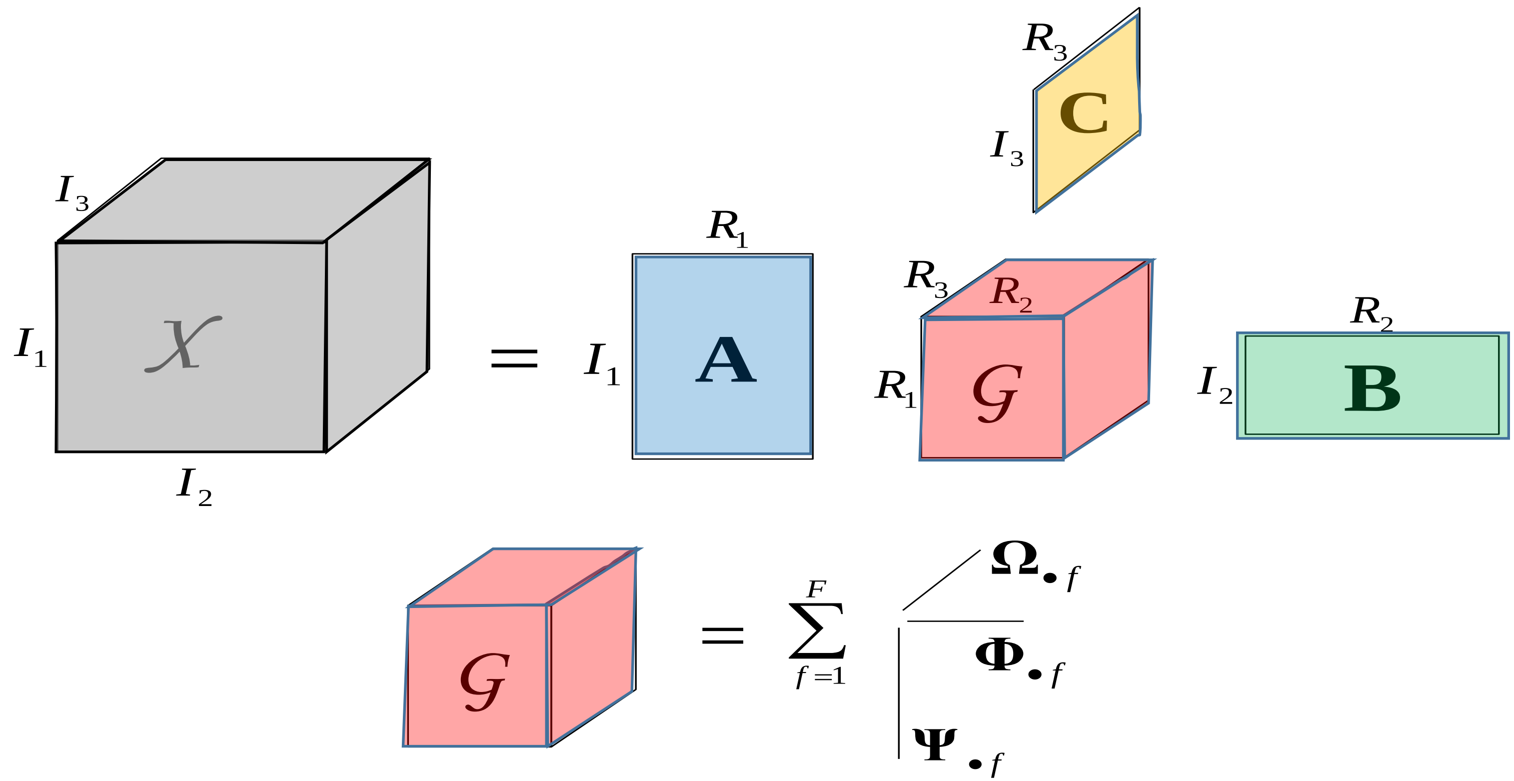
- *n*-mode product notation

$$\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{G}} \times_1 \boldsymbol{A} \times_2 \boldsymbol{B} \times_3 \boldsymbol{C}$$

- "Vectorized" form

$$\boldsymbol{x} = (\boldsymbol{A} \otimes \boldsymbol{B} \otimes \boldsymbol{C}) \boldsymbol{g}$$

# CONFAC decomposition




*CONFAC decomposition → equivalent to a constrained Tucker-3 decomposition with PARAFAC-decomposed core tensor in terms of (repeated) canonical vectors*

# CONFAC decomposition

- Scalar writing of $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$

$$x_{i_1,i_2,i_3} = \sum_{r_1=1}^{R_1} \sum_{r_2=1}^{R_2} \sum_{r_3=1}^{R_3} a_{i_1,r_1} b_{i_2,r_2} c_{i_3,r_3} g_{r_1,r_2,r_3}(\mathbf{\Psi}, \mathbf{\Phi}, \mathbf{\Omega}),$$

where $g_{r_1,r_2,r_3}(\mathbf{\Psi}, \mathbf{\Phi}, \mathbf{\Omega}) = \sum_{f=1}^{F} \psi_{r_1,f} \phi_{r_2,f} \omega_{r_3,f}$ with $F \geq \max(R_1, R_2, R_3)$

→ The columns of the constraint matrices $\mathbf{\Psi}$, $\mathbf{\Phi}$, and $\mathbf{\Omega}$ are canonical basis vectors (1's and 0's)

**PARAFAC:**

$$R_1 = R_2 = R_3 = F$$

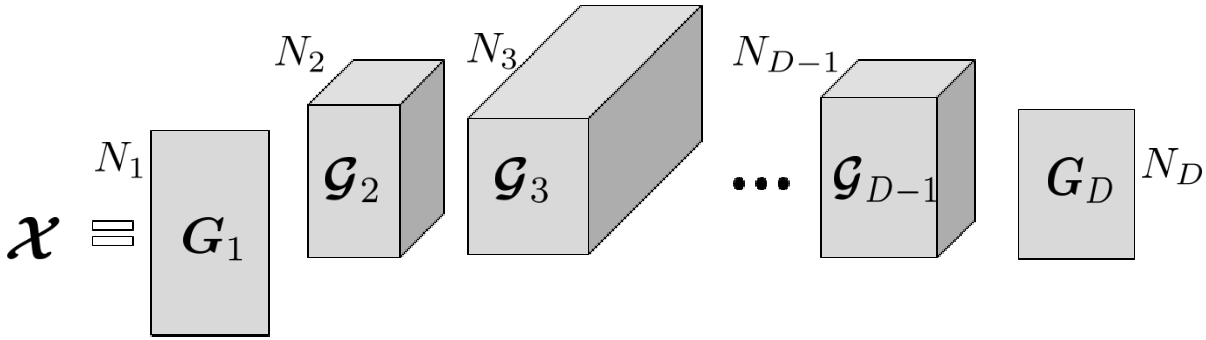
$$\mathbf{\Psi} = \mathbf{\Phi} = \mathbf{\Omega} = \mathbf{I}_Q$$

$$\mathcal{G}(\mathbf{\Psi}, \mathbf{\Phi}, \mathbf{\Omega}) = \mathcal{I}_Q$$

# Tensor-Train (TT) decomposition

- $D$-dimensional tensor as a "train" of smaller 3D tensors

[Oseledets'2011]




$$\boldsymbol{\mathcal{X}} = \boldsymbol{G}_1 \times_2^1 \boldsymbol{\mathcal{G}}_2 \times_3^1 \boldsymbol{\mathcal{G}}_3 \times_4^1 \cdots \times_{D-1}^1 \boldsymbol{\mathcal{G}}_{D-1} \times_D^1 \boldsymbol{G}_D$$

Overcome the curse of dimensionality (for "big" tensors)

# Channel modeling & estimation

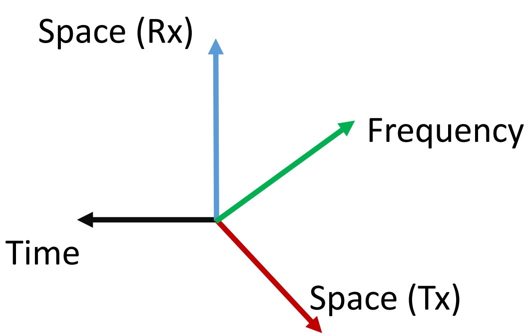
# “Tensorizing” the channel model

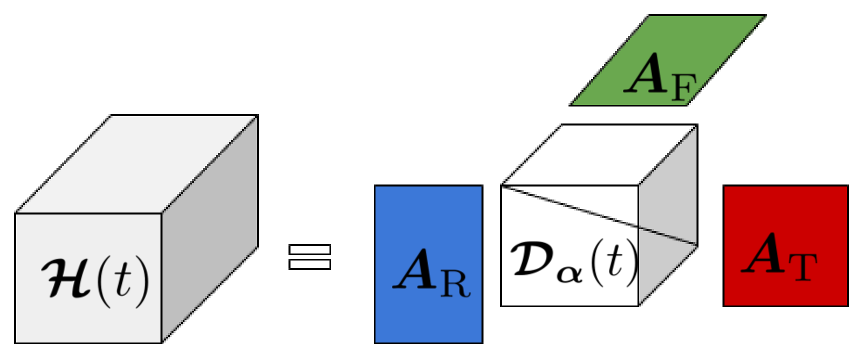
- Usual (matrix) notation

$$\boldsymbol{H}(t,f) = \sum_{\ell=1}^{N_{\mathrm{p}}} \alpha_\ell e^{j2\pi(\nu_\ell t - \tau_\ell f)} \boldsymbol{a}_{\mathrm{R}}(\theta_{R,\ell}, \phi_{R,\ell}) \boldsymbol{a}_{\mathrm{T}}^{*}(\theta_{T,\ell}, \phi_{T,\ell})$$

- Tensor notation (4D tensor, rank- $N_{\mathrm{p}}$ )

$$\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{D}}_{\boldsymbol{\alpha}} \times_1 \boldsymbol{A}_R(\boldsymbol{\theta}_{\mathrm{R}}, \boldsymbol{\phi}_{\mathrm{R}}) \times_2 \boldsymbol{A}_T(\boldsymbol{\theta}_{\mathrm{T}}, \boldsymbol{\phi}_{\mathrm{T}}) \times_3 \boldsymbol{A}_{\mathrm{D}}(\boldsymbol{\nu}) \times_4 \boldsymbol{A}_{\mathrm{F}}(\boldsymbol{\tau})$$




*Fixing the temporal mode → 3D tensor*

# “Tensorizing” the channel model (cont’d)

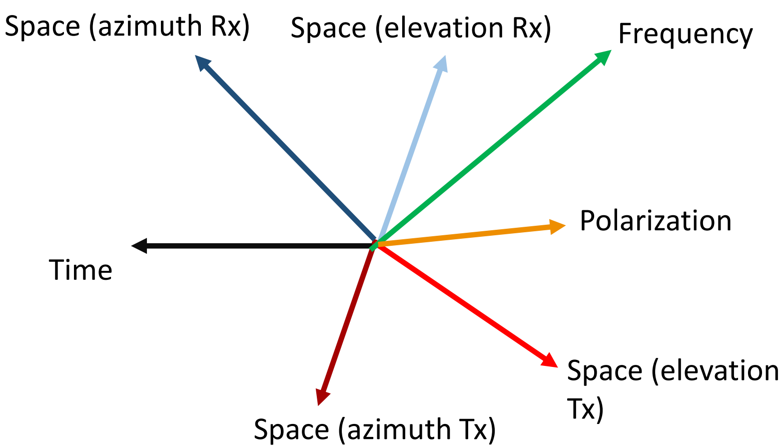
- Expanding the tensor (+ 2D antenna arrays, e.g. URA)

$$\boldsymbol{A}_{\mathrm{R}}(\boldsymbol{\theta}_{\mathrm{R}}, \boldsymbol{\phi}_{\mathrm{R}}) = \boldsymbol{A}_{\mathrm{R}}(\boldsymbol{\mu}_{\mathrm{R}}^{(1)}) \diamond \boldsymbol{A}_{\mathrm{R}}(\boldsymbol{\mu}_{\mathrm{R}}^{(2)}) \qquad \boldsymbol{A}_{\mathrm{T}}(\boldsymbol{\theta}_{\mathrm{T}}, \boldsymbol{\phi}_{\mathrm{T}}) = \boldsymbol{A}_{\mathrm{T}}(\boldsymbol{\mu}_{\mathrm{T}}^{(1)}) \diamond \boldsymbol{A}_{\mathrm{T}}(\boldsymbol{\mu}_{\mathrm{T}}^{(2)})$$

$$\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{D}}_{\boldsymbol{\alpha}} \times_1 \boldsymbol{A}_R(\boldsymbol{\mu}_{\mathrm{R}}^{(1)}) \times_2 \boldsymbol{A}_R(\boldsymbol{\mu}_{\mathrm{R}}^{(2)}) \times_3 \boldsymbol{A}_T(\boldsymbol{\mu}_{\mathrm{T}}^{(1)}) \times_4 \boldsymbol{A}_T(\boldsymbol{\mu}_{\mathrm{T}}^{(2)}) \times_5 \boldsymbol{A}_{\mathrm{D}}(\boldsymbol{\nu}) \times_6 \boldsymbol{A}_{\mathrm{F}}(\boldsymbol{\tau})$$

- Expanding the tensor (+ polarization) → 7 dimensions

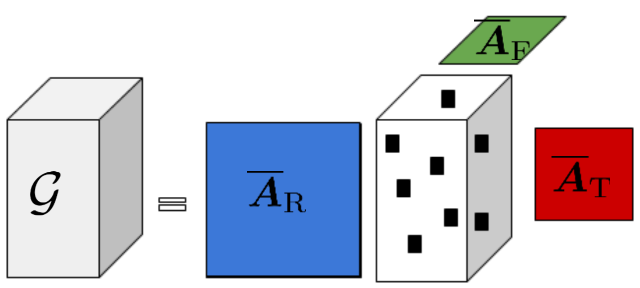
$$\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{D}}_{\boldsymbol{\alpha}} \times_1 \boldsymbol{A}_R(\boldsymbol{\mu}_{\mathrm{R}}^{(1)}) \times_2 \boldsymbol{A}_R(\boldsymbol{\mu}_{\mathrm{R}}^{(2)}) \times_3 \boldsymbol{A}_T(\boldsymbol{\mu}_{\mathrm{T}}^{(1)}) \times_4 \boldsymbol{A}_T(\boldsymbol{\mu}_{\mathrm{T}}^{(2)}) \times_5 \boldsymbol{A}_{\mathrm{D}}(\boldsymbol{\nu}) \times_6 \boldsymbol{A}_{\mathrm{F}}(\boldsymbol{\tau}) \times_7 \boldsymbol{B}_{\mathrm{pol}}$$

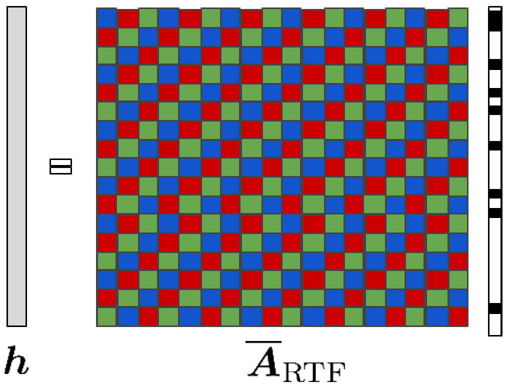



$$\boldsymbol{B}_{\mathrm{pol}} = \begin{bmatrix} \boldsymbol{b}^T(\mathrm{V}_r, \mathrm{V}_t) \\ \boldsymbol{b}^T(\mathrm{V}_r, \mathrm{H}_t) \\ \boldsymbol{b}^T(\mathrm{H}_r, \mathrm{V}_t) \\ \boldsymbol{b}^T(\mathrm{H}_r, \mathrm{H}_t) \end{bmatrix}$$

# Motivation (i.i.d chanels *or* structured channels?)

- Realistic channel models are not i.i.d →highly structured
- Algebraic channel structure is heterogeneous in different domains (e.g. space, frequency, time, polarization, etc...)
- Multidimensional channel structure is lost when working with vectorized (or "matricized") versions of the channel



# MIMO Channels & Compressed Sensing

Proceedings of the IEEE, vol. 98, no. 6, 2010

## Compressed Channel Sensing: A New Approach to Estimating Sparse Multipath Channels

*High-rate wireless data communication can usually be achieved by collecting a relatively small sample of the available information about the communications channel.*

By Waheed U. Bajwa, *Member IEEE*, Jarvis Haupt, Akbar M. Sayeed, *Senior Member IEEE*, and Robert Nowak, *Fellow IEEE*

IEEE JOURNAL OF SELECTED TOPICS IN SIGNAL PROCESSING, VOL. 10, NO. 3, 2016 1

## An Overview of Signal Processing Techniques for Millimeter Wave MIMO Systems

Robert W. Heath Jr., Nuria González-Prelcic, Sundeep Rangan, Wonil Roh, and Akbar Sayeed

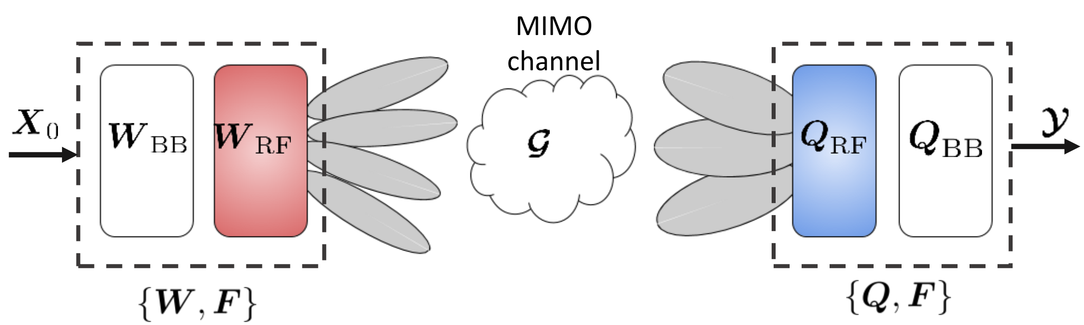
# Exploiting multilinearity & sparsity: *Tensor-CS*

IEEE Access, 2019

**Tensor-Based Channel Estimation for Massive MIMO-OFDM Systems**

Daniel C. Araújo, André L. F. de Almeida, João P. C. L. da Costa, Rafael T. de Sousa Jr.

- Hybrid A/D architecture

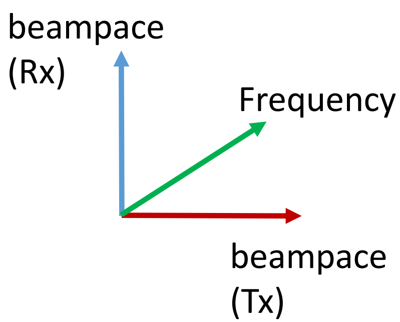


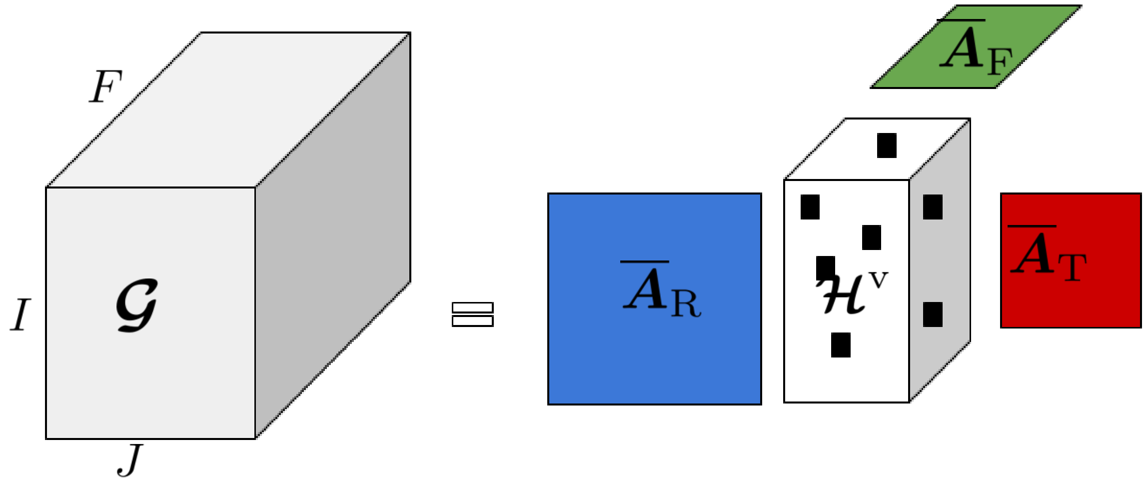

- "Compressed" measurement tensor:

$$\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{G}} \times_1 \underbrace{\boldsymbol{Q}}_{\text{Combiner (RF chains)}} \times_2 \underbrace{(\boldsymbol{X}_0 \boldsymbol{W})}_{\text{Precoded pilots}} \times_3 \underbrace{\boldsymbol{F}}_{\text{Subcarrier allocation}} + \tilde{\boldsymbol{\mathcal{Z}}}$$

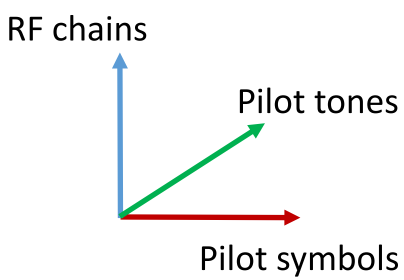
# Exploiting multilinearity & sparsity: *Tensor-CS*

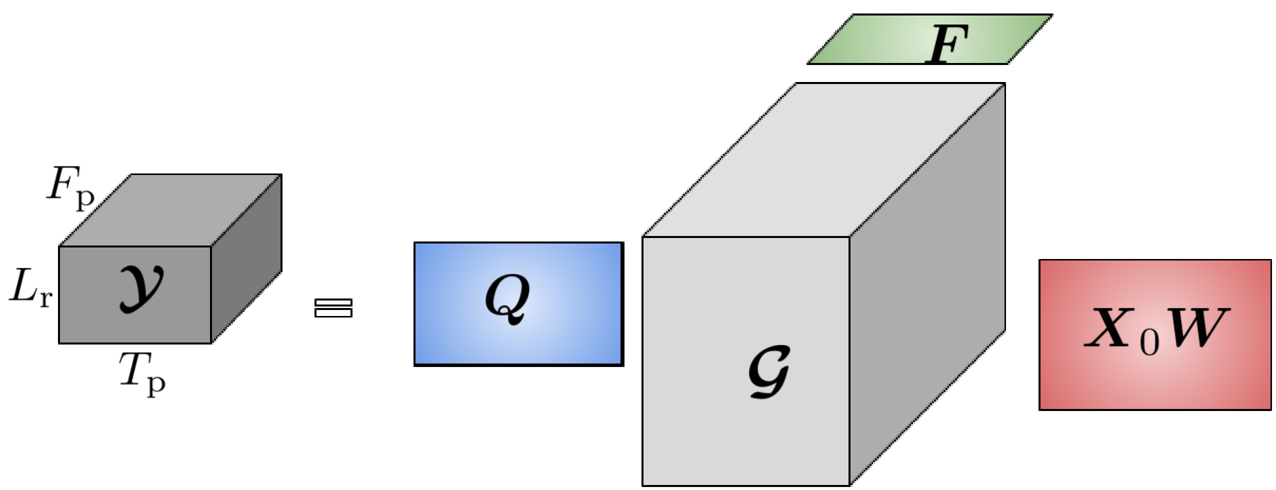
**Channel tensor (sparse Tucker3)**




**Compressed Rx signal tensor**



# Exploiting multilinearity & sparsity: *Tensor-CS*

- Expanding the 3D sparse channel tensor...

$$\mathcal{Y} = \mathcal{H}^{\mathrm{v}} \times_1 (\boldsymbol{Q}\overline{\boldsymbol{A}}_{\mathrm{R}}) \times_2 (\boldsymbol{X}_0\boldsymbol{W}\overline{\boldsymbol{A}}_{\mathrm{T}}) \times_3 (\boldsymbol{F}\overline{\boldsymbol{A}}_{\mathrm{F}}) + \tilde{\mathcal{Z}}$$

**Multi-linear compression !**

[Caiafa & Cichocki'2013]

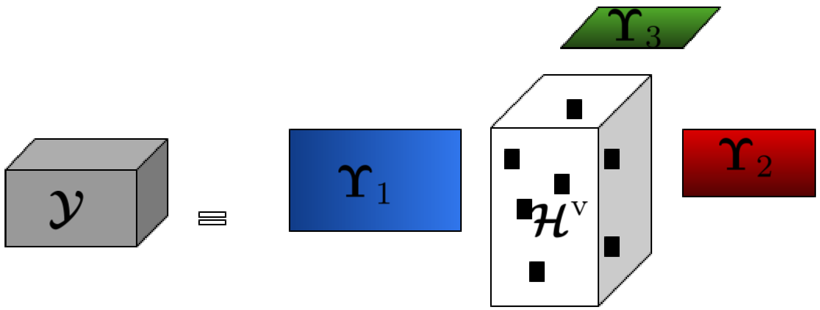
[Friedland, Li, Schonfeld '2014]

- Equivalent "vectorized" Kronecker- CS model [Duarte & Braniuk'2012]

$$\boldsymbol{y} = \left[(\boldsymbol{F}\overline{\boldsymbol{A}}_{\mathrm{F}}) \otimes (\boldsymbol{X}_0\boldsymbol{W}\overline{\boldsymbol{A}}_{\mathrm{T}}) \otimes (\boldsymbol{Q}\overline{\boldsymbol{A}}_{\mathrm{R}})\right]\boldsymbol{h}^{\mathrm{v}} + \tilde{\boldsymbol{z}}$$

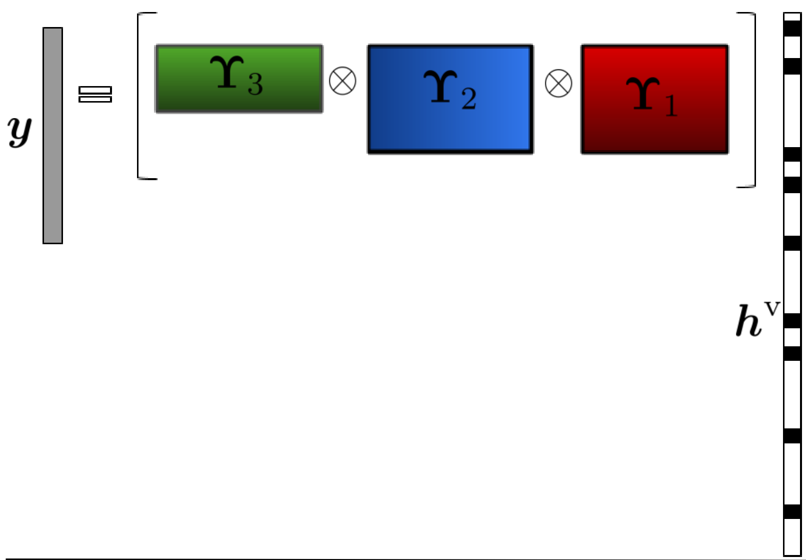
$$\boldsymbol{y} = \mathrm{vec}(\mathcal{Y}), \quad \boldsymbol{h}^{\mathrm{v}} = \mathrm{vec}(\mathcal{H}^{\mathrm{v}}), \quad \tilde{\boldsymbol{z}} = \mathrm{vec}(\tilde{\mathcal{Z}})$$

# Tensor-CS vs. Vector-CS




$$\mathrm{O}((L_{\mathrm{r}} + T_{\mathrm{p}} + F_{\mathrm{p}}))\log(L_{\mathrm{r}} + T_{\mathrm{p}} + F_{\mathrm{p}}) \leq C \leq \mathrm{O}(L_{\mathrm{r}}^3 + T_{\mathrm{p}}^3 + F_{\mathrm{p}}^3)$$

**Significant reduction on computational complexity**



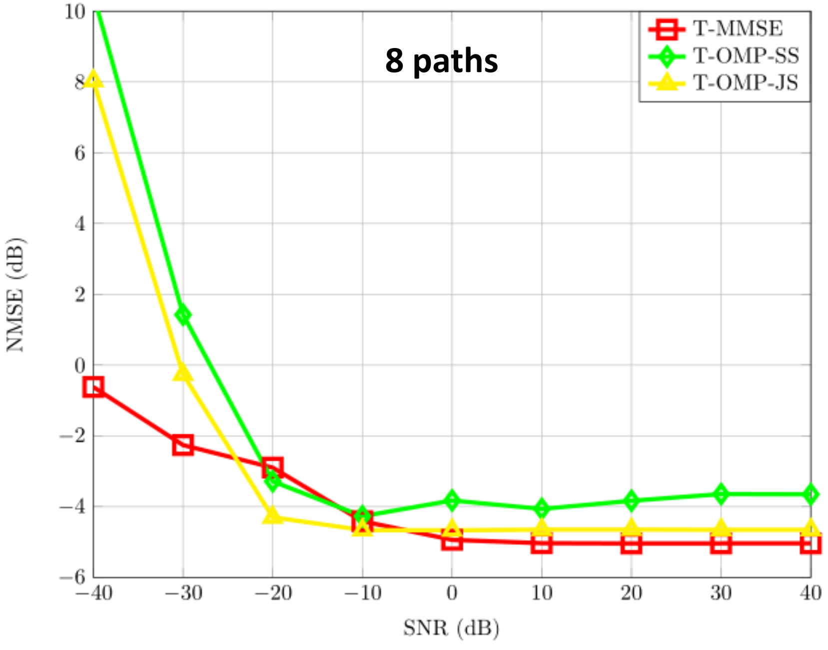

$$\mathrm{O}(L_{\mathrm{r}}T_{\mathrm{p}}F_{\mathrm{p}}\log(L_{\mathrm{r}}T_{\mathrm{p}}F_{\mathrm{p}}) \leq C \leq \mathrm{O}(L_{\mathrm{r}}^3T_{\mathrm{p}}^3F_{\mathrm{p}}^3)$$

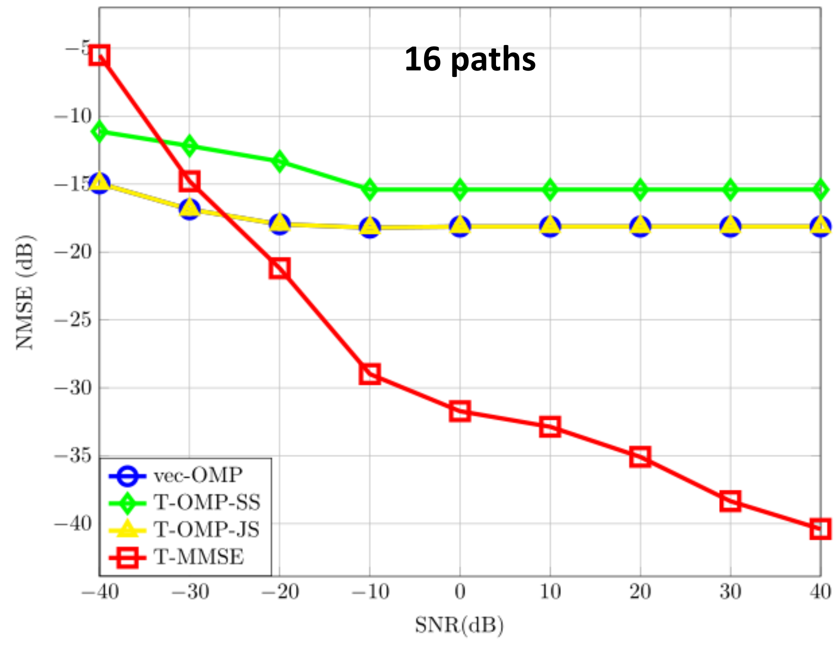
# Tensor-OMP algorithm

| | # antennas | # RF chains |
|---|---|---|
| Tx | 64 | 32 |
| Rx | 4 | 2 |

$$F = 1024, \quad F_{\mathrm{p}} = 256$$

$$T_{\mathrm{p}} = L_{\mathrm{t}} = 32$$



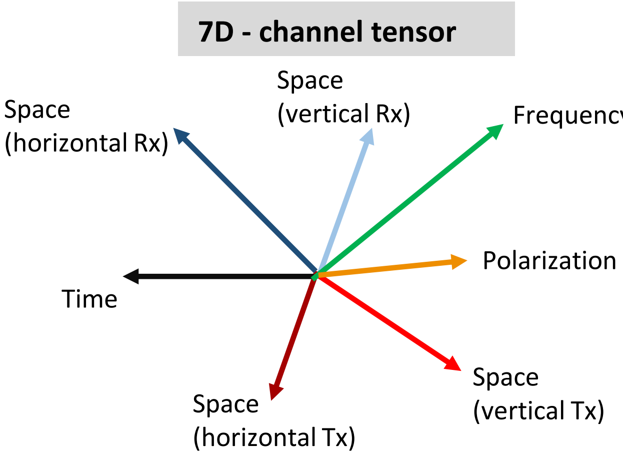
# *Tensor Train* Based Channel Estimation

- Time-frequency selective channel, URA, dual polarized antennas

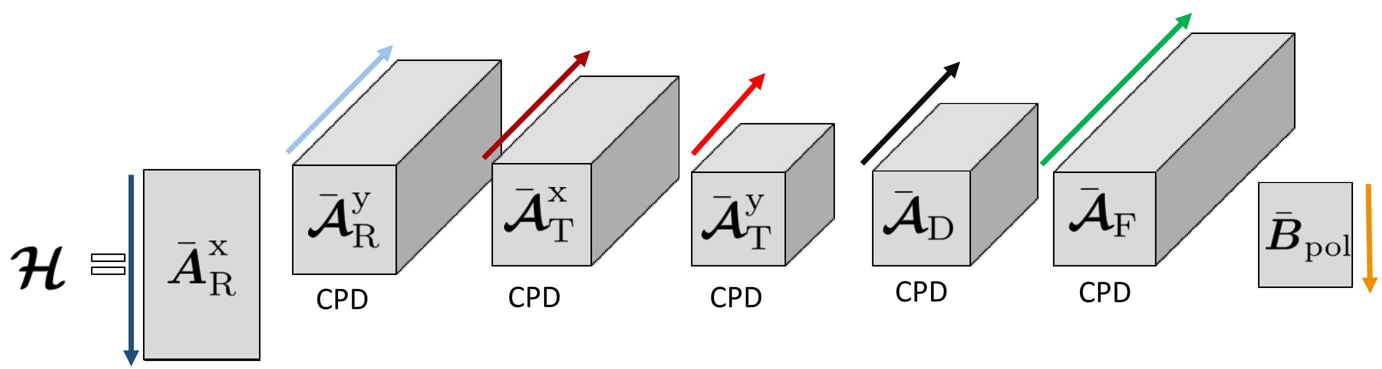
$$\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{I}}_{7,N_{\mathrm{p}}} \times_1 \boldsymbol{A}_{\mathrm{R}}^{(\mathrm{x})} \times_2 \boldsymbol{A}_{\mathrm{R}}^{(\mathrm{y})} \times_3 \boldsymbol{A}_{\mathrm{T}}^{(\mathrm{x})*} \times_4 \boldsymbol{A}_{\mathrm{T}}^{(\mathrm{y})*} \times_5 \boldsymbol{A}_{\mathrm{D}} \times_6 \boldsymbol{A}_{\mathrm{F}} \times_7 \boldsymbol{B}_{\mathrm{pol}}$$

**Example: 64 x 32 URA MIMO, T=10, F= 128, 4 polarization pairs**
**# coefficients: 10.485.760 → very large tensor!!**

**How to reduce complexity of channel representation and estimation ?**



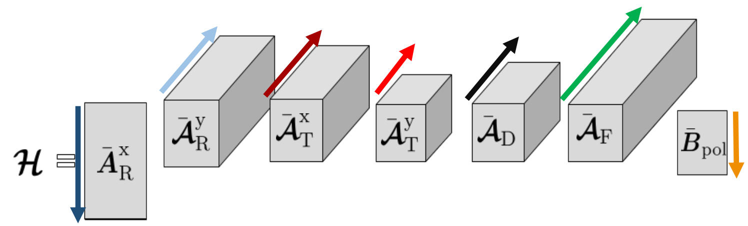

- Recasting the channel using Tensor Train model



# *Tensor Train* Massive MIMO Channel Modeling

Tensor Train Representation of Massive MIMO Channels using the Joint Dimensionality Reduction and Factor Retrieval (JIRAFE) Method

Yassine Zniyed, Rémy Boyer, *Senior Member, IEEE,* André L. F. de Almeida, *Senior Member, IEEE,* and Gérard Favier




## Dimensionality reduction

Tensor Train – SVD (TT-SVD) [Oseledets'2011]

$$[\bar{\boldsymbol{\mathcal{A}}}_{\mathrm{R}}^{(\mathrm{x})}, \bar{\boldsymbol{\mathcal{A}}}_{\mathrm{R}}^{(\mathrm{y})}, \bar{\boldsymbol{\mathcal{A}}}_{\mathrm{T}}^{(\mathrm{x})}, \bar{\boldsymbol{\mathcal{A}}}_{\mathrm{T}}^{(\mathrm{y})}, \bar{\boldsymbol{\mathcal{A}}}_{\mathrm{D}}, \bar{\boldsymbol{\mathcal{A}}}_{\mathrm{F}}, \bar{\boldsymbol{B}}_{\mathrm{pol}}] \leftarrow \text{TT-SVD}(\boldsymbol{\mathcal{H}}, N_{\mathrm{p}})$$

## Factors retrieval

Coupled LS optimization

$$F_{\mathrm{global}} = \sum_{i=1}^{7} F_i$$

CPD's

$$F_1 = |||\bar{\boldsymbol{A}}_{\mathrm{R}}^{(\mathrm{x})} - \boldsymbol{A}_{\mathrm{R}}^{(\mathrm{x})} \boldsymbol{M}_1^{-1}|||_F^2$$

$$F_2 = |||\bar{\boldsymbol{\mathcal{A}}}_{\mathrm{R}}^{(\mathrm{y})} - \boldsymbol{\mathcal{I}}_{3,N_{\mathrm{p}}} \times_1 \boldsymbol{M}_1 \times_2 \boldsymbol{A}_{\mathrm{R}}^{(\mathrm{y})*} \times_3 \boldsymbol{M}_2^{-T}|||_F^2$$

$$F_3 = |||\bar{\boldsymbol{\mathcal{A}}}_{\mathrm{T}}^{(\mathrm{x})} - \boldsymbol{\mathcal{I}}_{3,N_{\mathrm{p}}} \times_1 \boldsymbol{M}_2 \times_2 \boldsymbol{A}_{\mathrm{T}}^{(\mathrm{x})*} \times_3 \boldsymbol{M}_3^{-T}|||_F^2$$

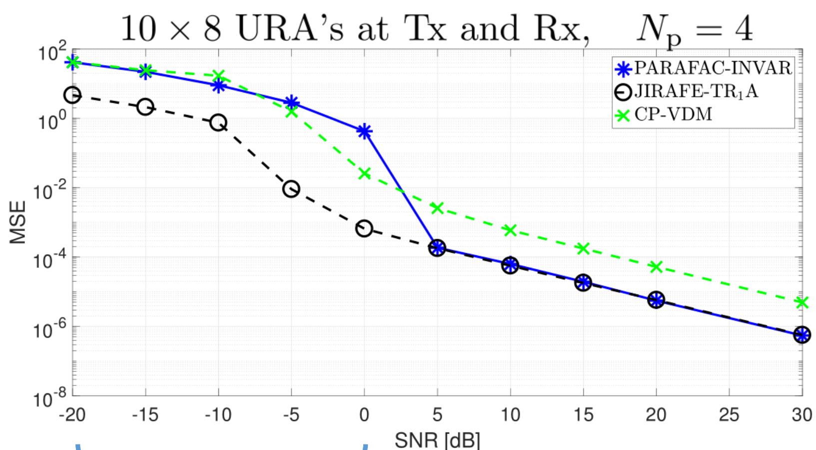
$$F_4 = |||\bar{\boldsymbol{\mathcal{A}}}_{\mathrm{T}}^{(\mathrm{y})} - \boldsymbol{\mathcal{I}}_{3,N_{\mathrm{p}}} \times_1 \boldsymbol{M}_3 \times_2 \boldsymbol{A}_{\mathrm{T}}^{(\mathrm{y})*} \times_3 \boldsymbol{M}_4^{-T}|||_F^2$$

$$F_5 = |||\bar{\boldsymbol{\mathcal{A}}}_{\mathrm{D}} - \boldsymbol{\mathcal{I}}_{3,N_{\mathrm{p}}} \times_1 \boldsymbol{M}_4 \times_2 \boldsymbol{A}_{\mathrm{D}} \times_3 \boldsymbol{M}_5^{-T}|||_F^2$$

$$F_6 = |||\bar{\boldsymbol{\mathcal{A}}}_{\mathrm{F}} - \boldsymbol{\mathcal{I}}_{3,N_{\mathrm{p}}} \times_1 \boldsymbol{M}_5 \times_2 \boldsymbol{A}_{\mathrm{F}} \times_3 \boldsymbol{M}_6^{-T}|||_F^2$$

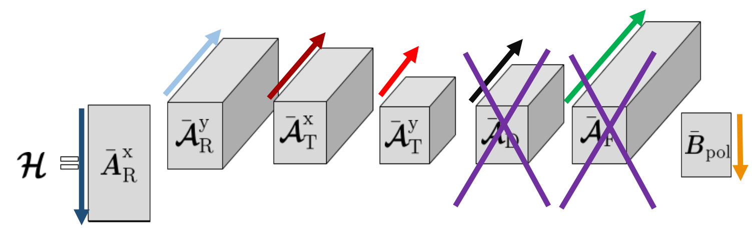
$$F_7 = |||\bar{\boldsymbol{B}}_{\mathrm{pol}} - \boldsymbol{M}_6 \boldsymbol{B}_{\mathrm{pol}}|||_F^2$$

# *Tensor Train* Massive MIMO Channel Modeling




Low SNR region

5-D channel tensor
(delay and Doppler not included)




PARAFAC-INVAR: [Qian, Fu, Sidiropoulos'2018]
CP-VDM: [Sørensen & De Lathauwer'2013]

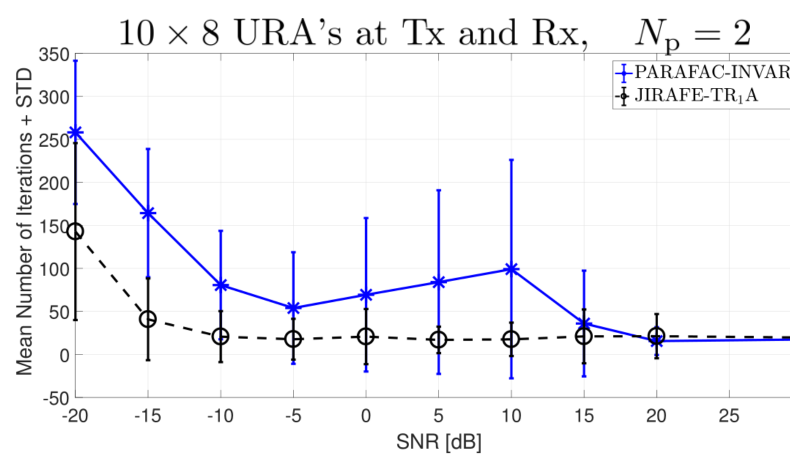
**Complexity reduction**

$$O(5N_p^2 M^4) \rightarrow O(3N_p^2 M^2)$$

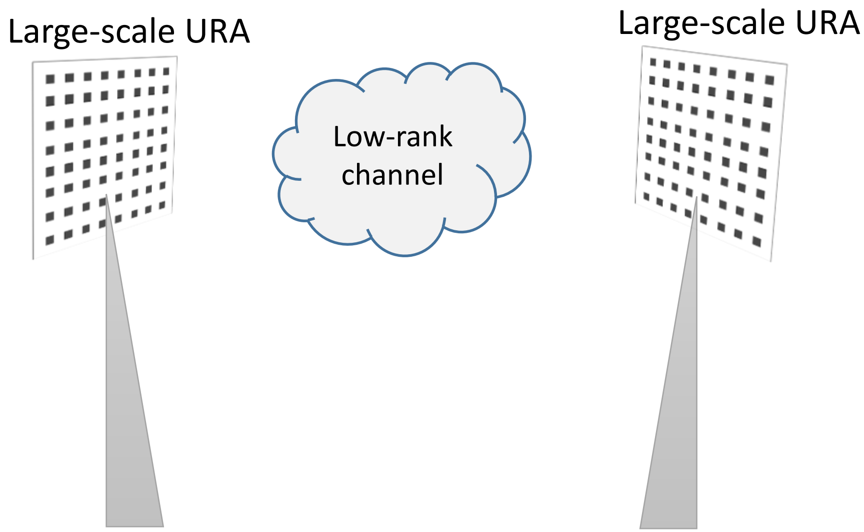
$$M = \max(M_R^{(x)}, M_R^{(y)}, M_T^{(x)}, M_T^{(y)})$$

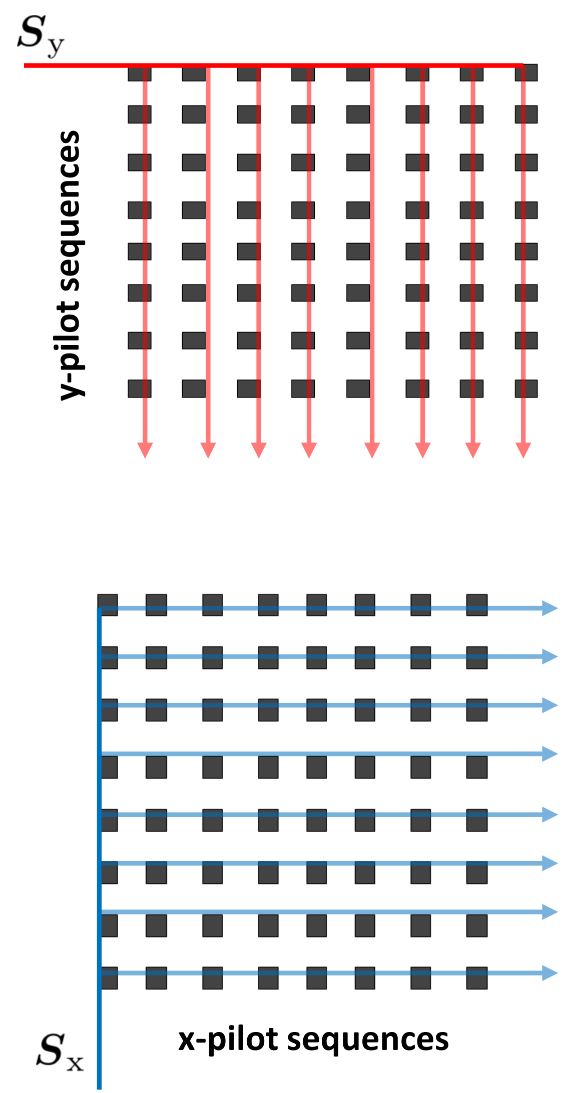
**JIRAFE approach**

- **More accurate estimates for low SNR's**
- **Faster convergence & reduced complexity than 5-order CPD fitting algorithms**
- **Gains should be higher for 7-D channel**



# *TensorPilots:* Kronecker-structured pilot schemes




- Rank-1 channel

$$\boldsymbol{H} = (\boldsymbol{h}_{\mathrm{R}}^{(\mathrm{x})} \otimes \boldsymbol{h}_{\mathrm{R}}^{(\mathrm{y})})(\boldsymbol{h}_{\mathrm{T}}^{(\mathrm{x})} \otimes \boldsymbol{h}_{\mathrm{T}}^{(\mathrm{y})})^{H}$$

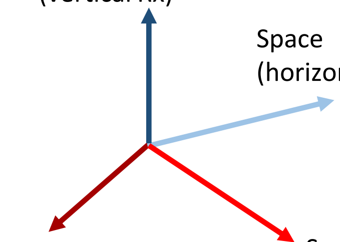
# *TensorPilots:* Kronecker-structured pilot schemes

- Exploting separability & sparsity at both link ends

$$\boldsymbol{X} = \underbrace{(\boldsymbol{Q}_{\mathrm{x}} \otimes \boldsymbol{Q}_{\mathrm{y}})}_{\text{combiners}}\boldsymbol{H}\underbrace{(\boldsymbol{S}_{\mathrm{x}} \otimes \boldsymbol{S}_{\mathrm{y}})}_{\text{pilots}} \qquad \boldsymbol{H} = (\boldsymbol{h}_{\mathrm{R}}^{(\mathrm{x})} \otimes \boldsymbol{h}_{\mathrm{R}}^{(\mathrm{y})})(\boldsymbol{h}_{\mathrm{T}}^{(\mathrm{x})} \otimes \boldsymbol{h}_{\mathrm{T}}^{(\mathrm{y})})^{H}$$

- Received pilot signals: rank-1 tensor

$$\boldsymbol{\mathcal{X}} = \underbrace{(\boldsymbol{Q}_{\mathrm{x}}\boldsymbol{h}_{\mathrm{R}}^{(\mathrm{x})})}_{\boldsymbol{z}_{\mathrm{R}}^{(\mathrm{x})}} \circ \underbrace{(\boldsymbol{Q}_{\mathrm{y}}\boldsymbol{h}_{\mathrm{R}}^{(\mathrm{y})})}_{\boldsymbol{z}_{\mathrm{R}}^{(\mathrm{y})}} \circ \underbrace{(\boldsymbol{S}_{\mathrm{x}}\boldsymbol{h}_{\mathrm{T}}^{(\mathrm{x})})}_{\boldsymbol{z}_{\mathrm{T}}^{(\mathrm{x})}} \circ \underbrace{(\boldsymbol{S}_{\mathrm{y}}\boldsymbol{h}_{\mathrm{T}}^{(\mathrm{y})})}_{\boldsymbol{z}_{\mathrm{T}}^{(\mathrm{y})}}$$




**Channel estimation with *TensorPilots***

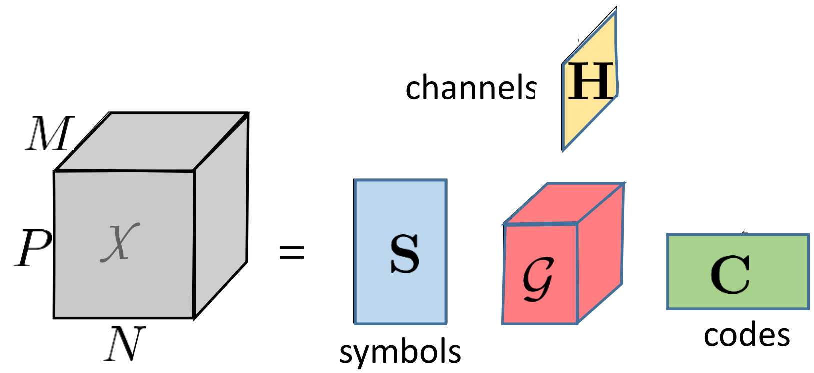
1. **Truncated Higher-Order SVD →** $\{\hat{\boldsymbol{z}}_{\mathrm{R}}^{(\mathrm{x})}, \hat{\boldsymbol{z}}_{\mathrm{R}}^{(\mathrm{y})}, \hat{\boldsymbol{z}}_{\mathrm{T}}^{(\mathrm{x})}, \hat{\boldsymbol{z}}_{\mathrm{T}}^{(\mathrm{y})}\}$
2. **Per-mode LS or CS recovery →** $\{\hat{\boldsymbol{\theta}}_{\mathrm{R}}, \hat{\boldsymbol{\phi}}_{\mathrm{R}}, \hat{\boldsymbol{\theta}}_{\mathrm{T}}, \hat{\boldsymbol{\phi}}_{\mathrm{T}}\}$

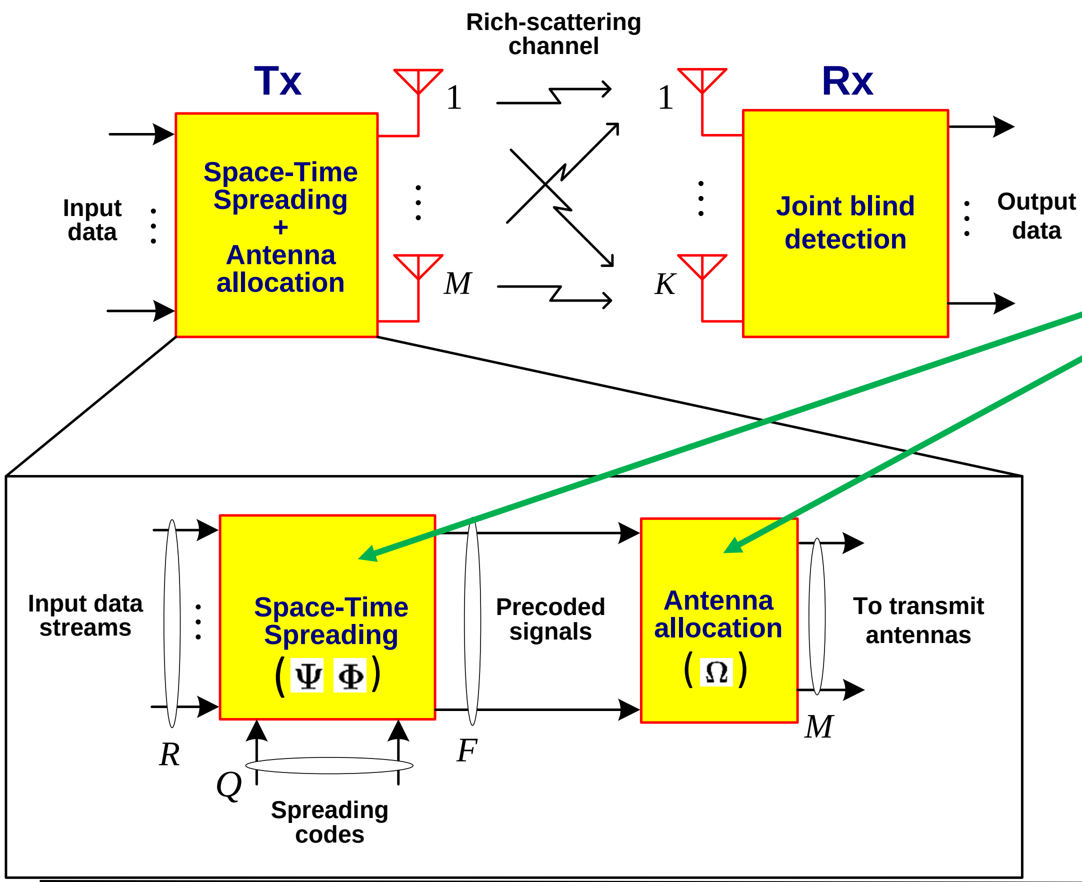
# Space-time-frequency (STF) MIMO

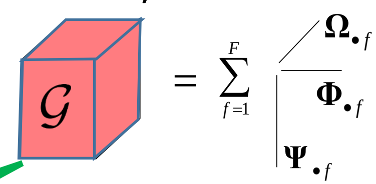
# CONFAC based MIMO transceivers

**Idea** [de Almeida and Favier, 2008]

- Design flexible schemes for ST-MIMO signaling
- Capitalize on the CONFAC uniqueness properties at the receiver to jointly recover the channel and symbols




$\Psi$ ⇨ *stream allocation* ($R$ x $F$)

$\Phi$ ⇨ *code allocation* ($Q$ x $F$)

$\Omega$ ⇨ *antenna allocation* ($M$ x $F$)

Designing tensor $\mathcal{G}$ defines the transmission scheme!

# CONFAC based MIMO transceivers

## Key features

- **Variable antenna allocation patterns:** Multiple data streams per transmit antenna
- **Variable spreading code reuse patterns:** Spreading codes can be reused by TX antennas
- **Transmission flexibility:** Several schemes possible by adjusting the allocation matrices

- Received signal ($n$-th symbol, $p$-th chip, $k$-th Rx antenna):

$$x_{k,n,p} = \sum_{m=1}^{M} \sum_{r=1}^{R} s_{n,r} c_{p,q} h_{k,m} \underbrace{g_{r,q,m}(\boldsymbol{\Psi}, \boldsymbol{\Phi}, \boldsymbol{\Omega})}_{\text{Resource allocation tensor}}$$

with $F \geq \max(R, Q, M)$

**PARAFAC DS-CDMA model**

[Sidiropoulos et al, 2000]

$$\boldsymbol{\Psi} = \boldsymbol{\Phi} = \boldsymbol{\Omega} = \mathbf{I}_F$$
$$\mathcal{G}(\boldsymbol{\Psi}, \boldsymbol{\Phi}, \boldsymbol{\Omega}) = \mathcal{I}_F$$

**Note:** columns of $\boldsymbol{\Psi}$, $\boldsymbol{\Phi}$, and $\boldsymbol{\Omega}$ are canonical basis vectors (1's and 0's)

# CONFAC based MIMO transceivers

**Example** (*R*=3 streams, *Q*=3 codes, *M*=2 TX antenas, *F*=4 coded signals)

$$\boldsymbol{\Psi} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad \boldsymbol{\Phi} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \boldsymbol{\Omega} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

stream #1, code #2, antenna #1

stream #2, code #3, antenna #2

**Note:** Allocations can be optimized

# CONFAC based MIMO transceivers

- Unfolded representations ("constrained-CP" writing)

$$\mathbf{X}_1 = \left(\mathbf{C\Phi} \diamond \mathbf{H\Omega}\right)\left(\mathbf{S\Psi}\right)^T, \quad \mathbf{X}_2 = \left(\mathbf{H\Omega} \diamond \mathbf{S\Psi}\right)\left(\mathbf{C\Phi}\right)^T$$

$$\mathbf{X}_3 = \left(\mathbf{S\Psi} \diamond \mathbf{C\Phi}\right)\left(\mathbf{H\Omega}\right)^T$$

- Partial uniqueness properties:
  - Symbol-only recovery (only $\mathbf{S}$ is unique)
  - Channel-only recovery (only $\mathbf{H}$ is unique)
  - Joint symbol-channel recovery (both $\mathbf{S}$ and $\mathbf{H}$ are unique)

Essential uniqueness result [Stegeman & de Almeida, 2009]

**Assumptions:** $\mathbf{S}, \mathbf{C}, \mathbf{H}$ full column rank; $\left(\mathbf{\Phi} \diamond \mathbf{\Omega}\right)\mathbf{\Psi}^T$ full column rank

Let $N^* = \max_r \left(\mathrm{rank}(\mathbf{\Phi}\mathrm{diag}(\psi_r^T)\mathbf{\Phi}^T)\right)$

If $\mathrm{rank}(\mathbf{\Phi}\mathrm{diag}(\mathbf{\Psi}^T\mathbf{d})\mathbf{\Phi}^T)\Big) \leq N^*$ implies $\omega(\mathbf{d}) \leq 1$ ⇒ $\mathbf{S}$ is unique

# CONFAC based MIMO transceivers

## CONFAC scheme *versus* PARAFAC scheme (KRST coding)

*CONFAC scheme (R=2, Q=2, M=4):*

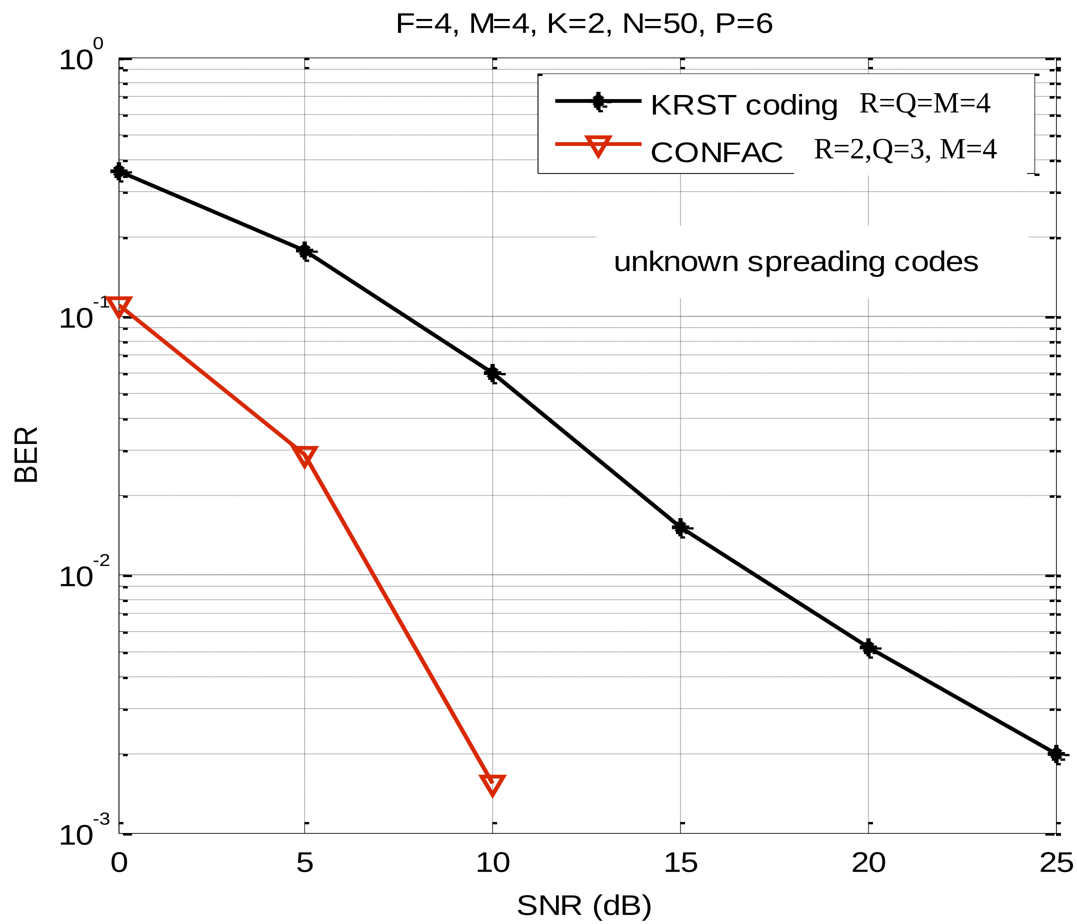
$$\boldsymbol{\Psi} = \boldsymbol{\Phi} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{\Omega} = \mathbf{I}_4$$

*KRST coding:*

$$\boldsymbol{\Psi} = \boldsymbol{\Phi} = \boldsymbol{\Omega} = \mathbf{I}_4$$

→ *Both schemes:*
*Blind detection **without the knowledge of the spreading codes***

→ *CONFAC scheme:*
*Lower BER than the KRST coding scheme (trade-off: rate reduction by a factor of 2)*



# Tensor Space-Time-Frequency (T-STF) Coding

**Idea** [de Almeida and Favier, 2014]

- Design generalized STF coding scheme with **allocation flexibility** over different STF domains (MIMO-OFDM-CDMA)

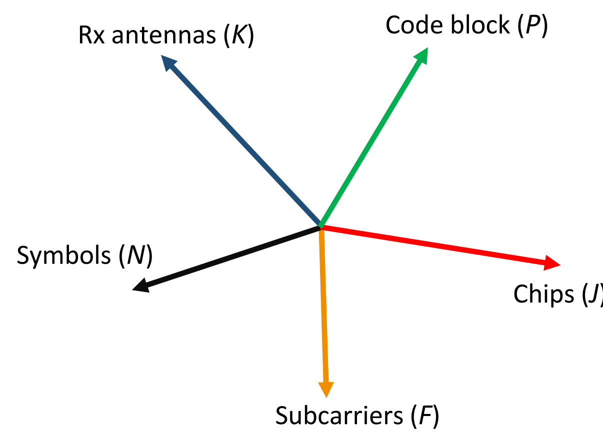
- T-STF coding model (5D)

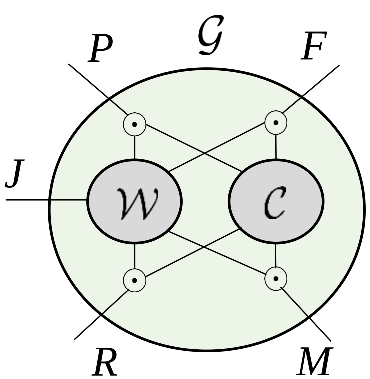
$$x_{k,n,f,p,j} = \sum_{m=1}^{M} \sum_{r=1}^{R} \underbrace{g_{m,r,f,p,j}}_{\text{Code tensor}} \underbrace{h_{k,m,f}}_{\text{Channel tensor}} \underbrace{s_{n,r}}_{\text{Symbol matrix}}$$

- Received signal (noiseless case)

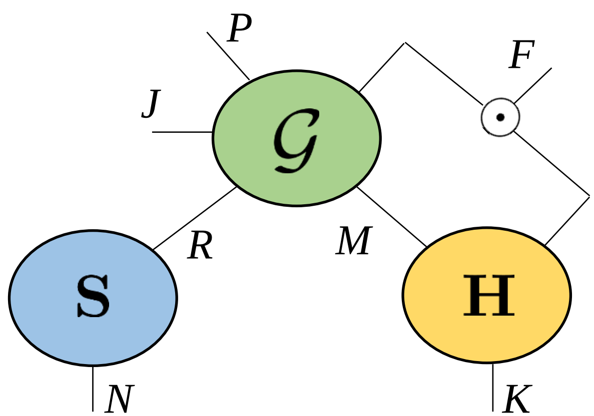
$$\mathcal{X} = \mathcal{G} \times_1 \mathcal{H} \times_2 \mathbf{S}$$ → Tucker-(2-5) model

with $\mathcal{G} = \underbrace{\mathcal{W}}_{\text{spreading tensor}} \underset{\{m,r,f,p\}}{\odot} \underbrace{\mathcal{C}}_{\text{allocation tensor}}$



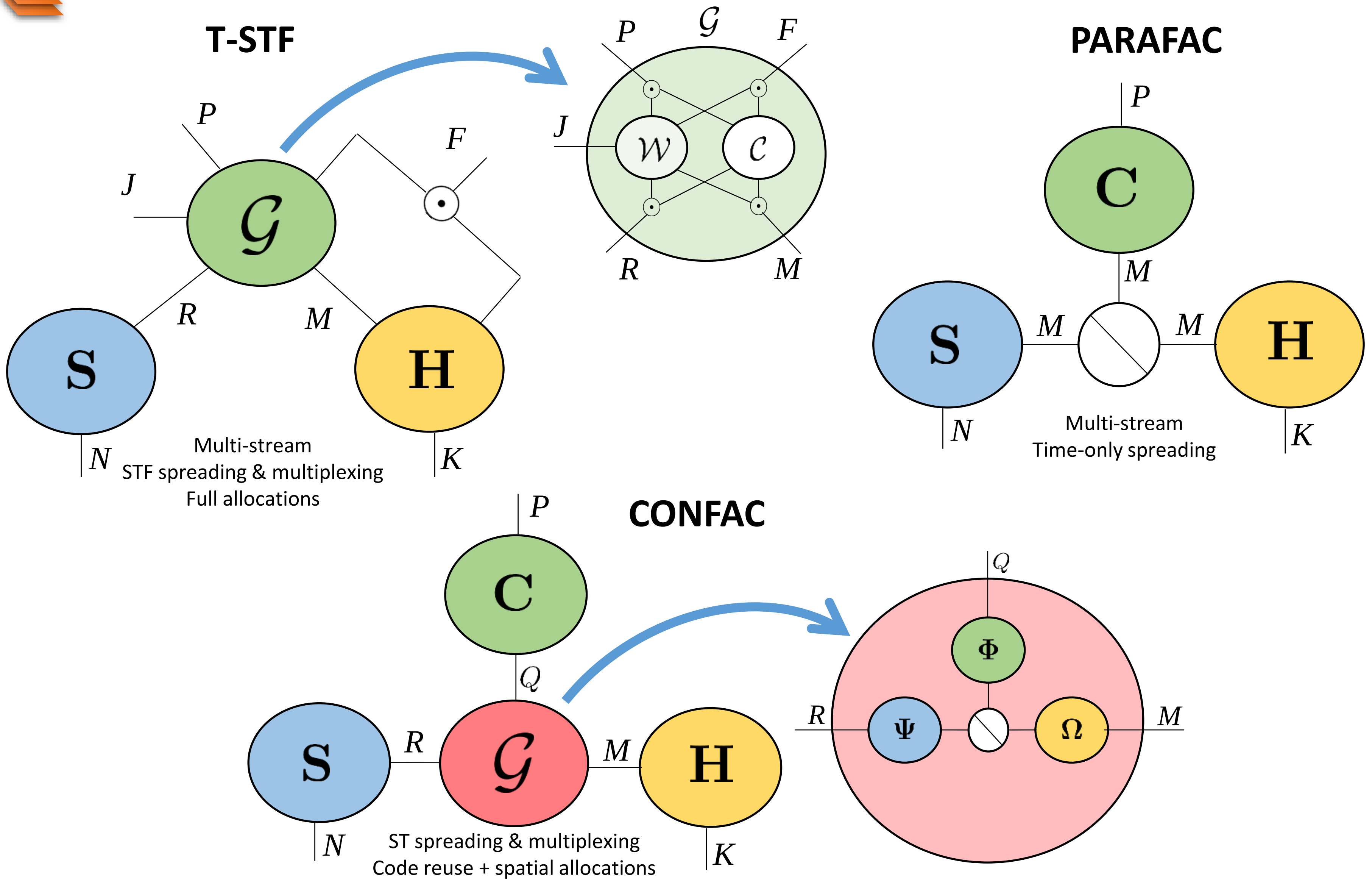



Code tensor diagram



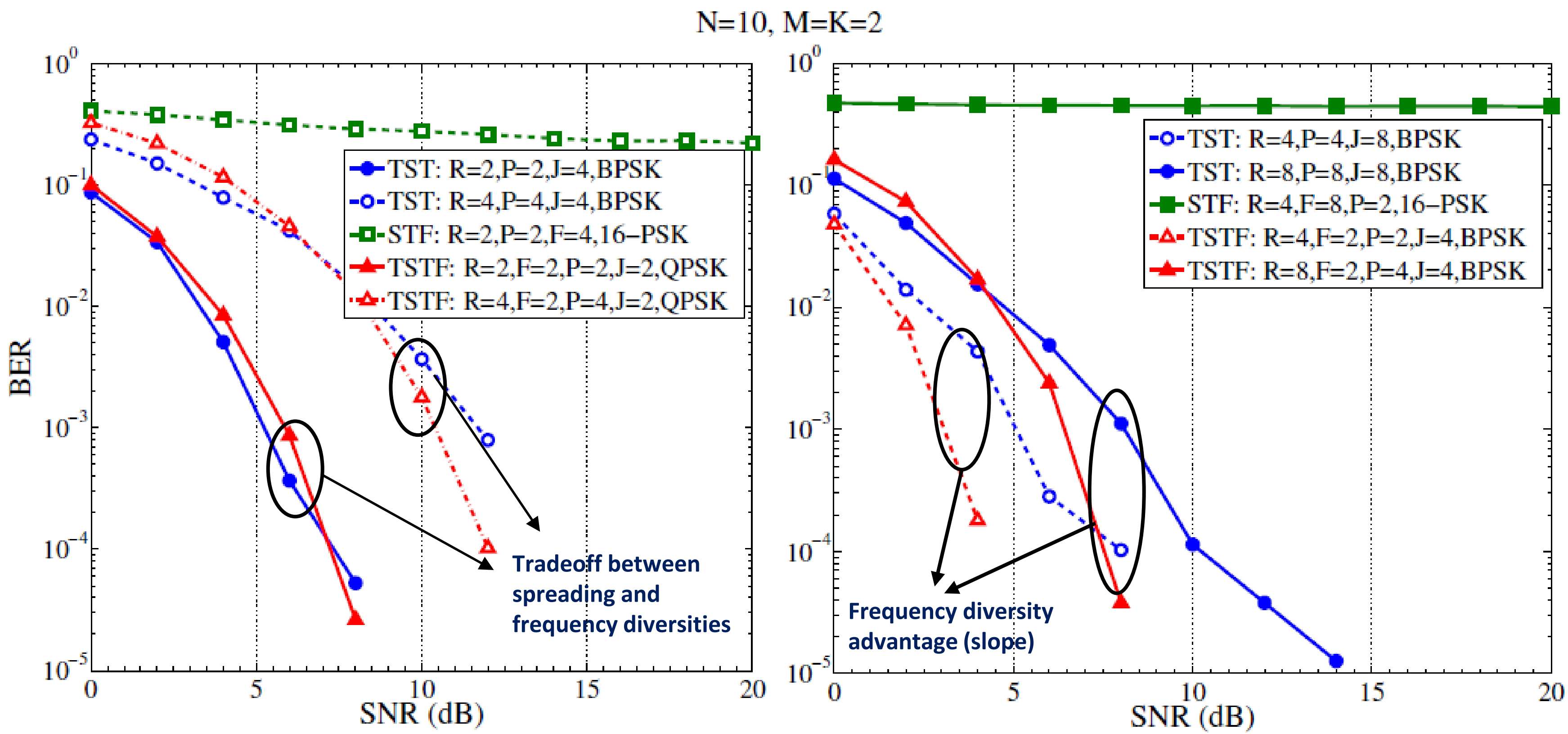
Received signal tensor diagram

# T-STF vs. CONFAC vs. PARAFAC schemes



# T-STF performance

## Random allocations, Kronecker-ALS algorithm




TSTF @ $F$=2 subcarriers

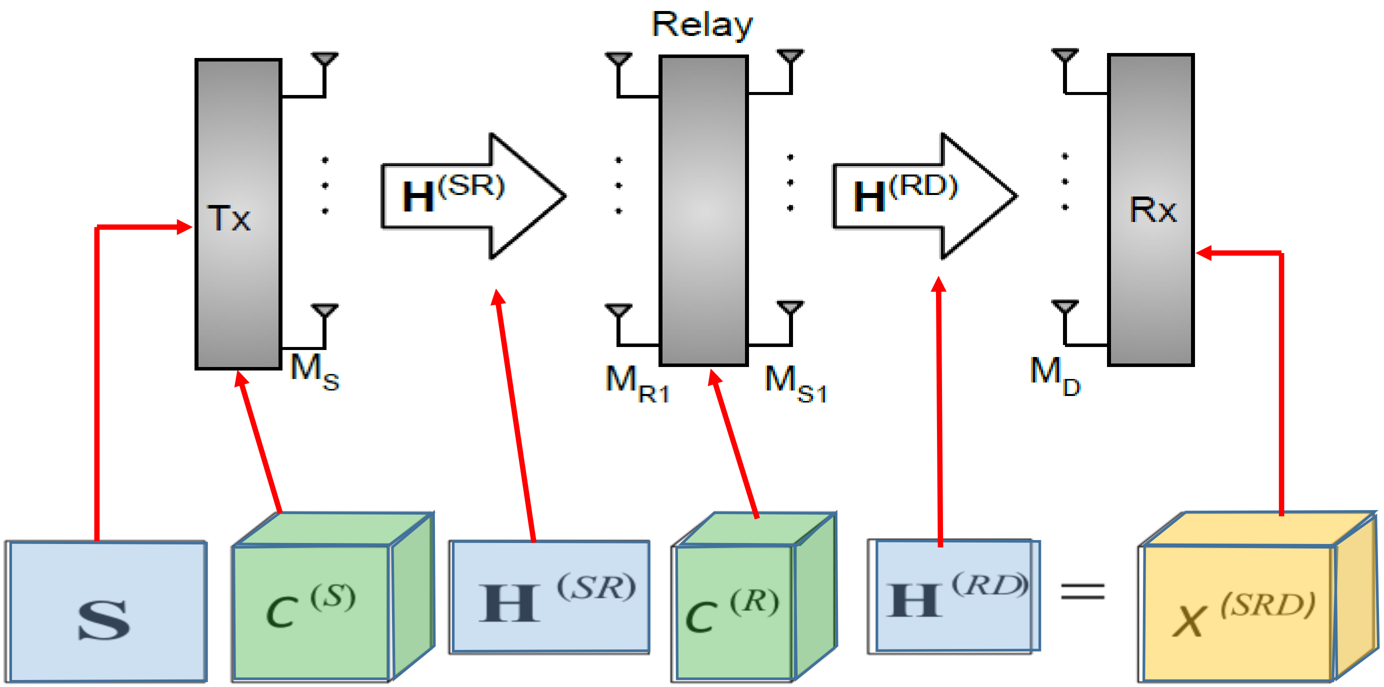
TSTF @ $F$=4 subcarriers

# Relaying-based communications

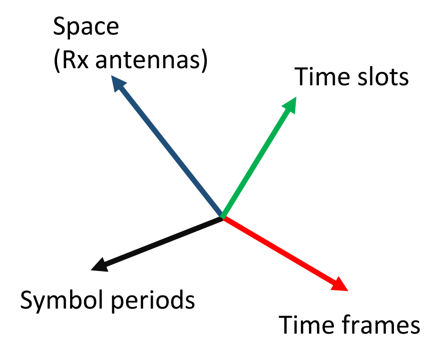
# Tensor-based MIMO Relaying

**Idea:** Use tensor coding at source and relay to jointly estimate the involved channels (source-relay and relay-destination)

[Ximenes et al, 2015]
[Fernandes et al, 2016]
[Znyed et al, 2018]
[Sokal et al, 2020]




**Nested Tucker-(2,4) model**




$$\mathcal{X}^{(\mathrm{SRD})} = (\mathcal{C}^{(S)} \times_2^1 \mathbf{H}^{(\mathrm{SR})} \times_2^1 \mathcal{C}^{(R)}) \times_1 \mathbf{S} \times_2 \mathbf{H}^{(\mathrm{RD})}$$

# Tensor-based MIMO Relaying

- Processing steps

$$\mathcal{X}^{(\mathrm{SRD})} = (\mathcal{C}^{(S)} \times_2^1 \mathbf{H}^{(\mathrm{SR})} \times_2^1 \mathcal{C}^{(R)}) \times_1 \mathbf{S} \times_2 \mathbf{H}^{(\mathrm{RD})}$$

Unfolding

$$[\mathcal{X}^{(\mathrm{RD})}]_{(1,2);(3,4)} = \left[\mathtt{vec}(\mathbf{H}^{(\mathrm{SR})})^{\mathtt{T}} \otimes \mathbf{H}^{(\mathrm{RD})} \otimes \mathbf{S}\right] \mathbf{Z}_{(\mathcal{C}^S, \mathcal{C}^R)}$$

Filtering $\mathbf{Z}^{\dagger}_{(\mathcal{C}^S, \mathcal{C}^R)}$

$$\mathbf{Y} \approx \mathtt{vec}(\mathbf{H}^{(\mathrm{SR})})^{\mathtt{T}} \otimes \mathbf{H}^{(\mathrm{RD})} \otimes \mathbf{S}$$

Kronecker approximation problem

$$\min \left\|\mathbf{Y} - \mathtt{vec}(\mathbf{H}^{(\mathrm{SR})})^{\mathtt{T}} \otimes \mathbf{H}^{(\mathrm{RD})} \otimes \mathbf{S}\right\|_F$$

Recast as a rank-1 tensor approximation problem (→ classical algorithms)

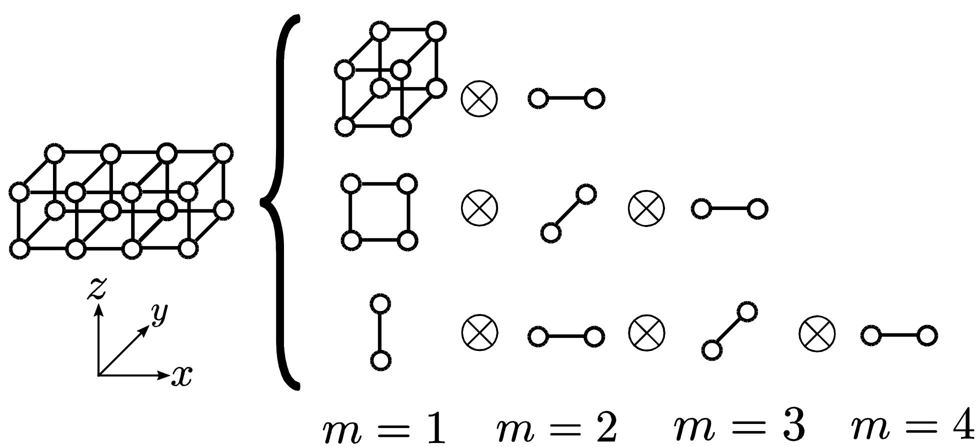
$$\min \left\|\mathcal{Y} - \mathbf{h}^{(\mathrm{SR})} \circ \mathbf{h}^{(\mathrm{RD})} \circ \mathbf{s}\right\|_F \rightarrow (\hat{\mathbf{h}}^{(\mathrm{SR})}, \hat{\mathbf{h}}^{(\mathrm{RD})}, \hat{\mathbf{s}})$$

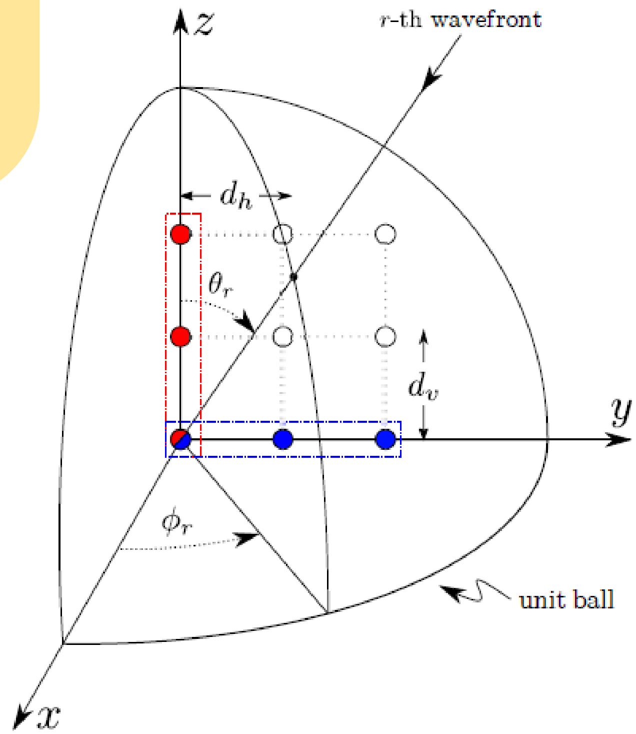
# Multi-linear beamforming

# Why multi-linear beamforming?

- As the size of a sensor array grows, the beamforming operation needs more...
  - ❖ Samples to estimate statistics
  - ❖ Computation time to obtain weights

- **Idea**: Exploit the algebraic structure of <u>separable arrays</u> → multi-linearity property



# Separable Arrays

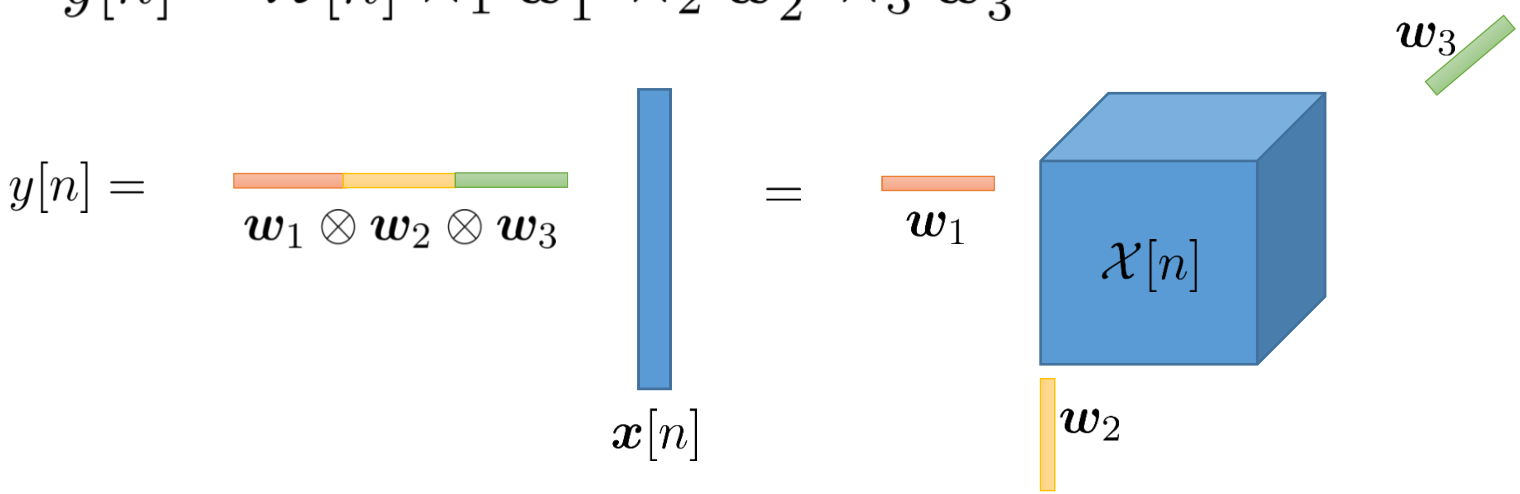
**Idea:** Kronecker filters as multilinear maps!

- Consider the trilinear filter:

$$y[n] = \boldsymbol{w}^{\mathsf{H}}\boldsymbol{x}[n] = (\boldsymbol{w}_1 \otimes \boldsymbol{w}_2 \otimes \boldsymbol{w}_3)^{\mathsf{H}}\boldsymbol{x}[n]$$

- Reshape the input signal vector into a 3d tensor:

$$y[n] = \mathcal{X}[n] \times_1 \boldsymbol{w}_1^{\mathsf{H}} \times_2 \boldsymbol{w}_2^{\mathsf{H}} \times_3 \boldsymbol{w}_3^{\mathsf{H}}$$



# Separable Arrays

- From tensor algebra, the trilinear filter output can be written as

$$
\begin{aligned}
y[n] &= \boldsymbol{w}_1^{\mathsf{H}} \boldsymbol{X}_{(1)}[n] (\boldsymbol{w}_3 \otimes \boldsymbol{w}_2)^* = \boldsymbol{w}_1^{\mathsf{H}} \boldsymbol{u}_1[n] \\
&= \boldsymbol{w}_2^{\mathsf{H}} \boldsymbol{X}_{(2)}[n] (\boldsymbol{w}_3 \otimes \boldsymbol{w}_1)^* = \boldsymbol{w}_2^{\mathsf{H}} \boldsymbol{u}_2[n] \\
&= \boldsymbol{w}_3^{\mathsf{H}} \boldsymbol{X}_{(3)}[n] (\boldsymbol{w}_2 \otimes \boldsymbol{w}_1)^* = \boldsymbol{w}_3^{\mathsf{H}} \boldsymbol{u}_3[n]
\end{aligned}
$$

Keep fixed

Linear w.r.t. each subfilter

**Main idea**:

- Design each “subfilter” instead of full filter
- Computational complexity reduction

# Tensor Beamforming Algorithms

- Alternating optimization approaches
  - Tensor LMS [Rupp & Schwarz'2015]
  - Tensor GSC [Miranda et al'2015]
  - **Tensor MMSE** [Ribeiro et al'2016, Ribeiro et al'2019]
  - **Tensor LCMV** [Ribeiro et al'2019]
  - **Tensor Frost** [Ribeiro et al'2019]

$N$-dimensional filter with $N = N_1 N_2 N_3$

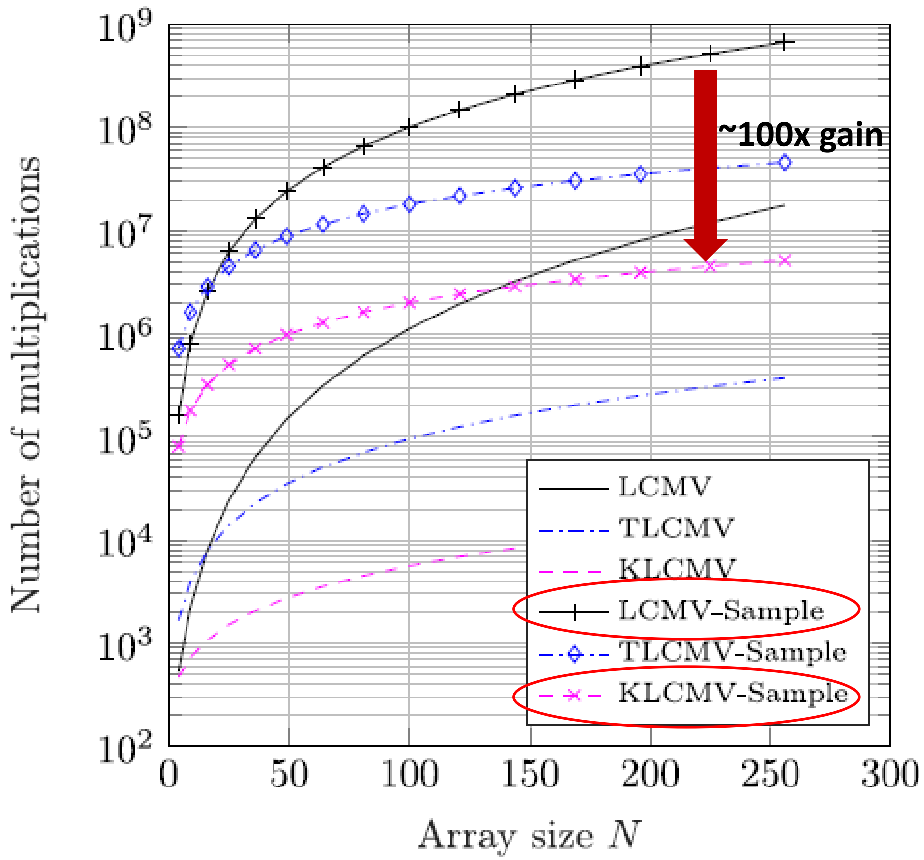
- <u>Example</u>: Trilinear filter design $\boldsymbol{w} = \underbrace{\boldsymbol{w}_1}_{N_1} \otimes \underbrace{\boldsymbol{w}_2}_{N_2} \otimes \underbrace{\boldsymbol{w}_3}_{N_3}$

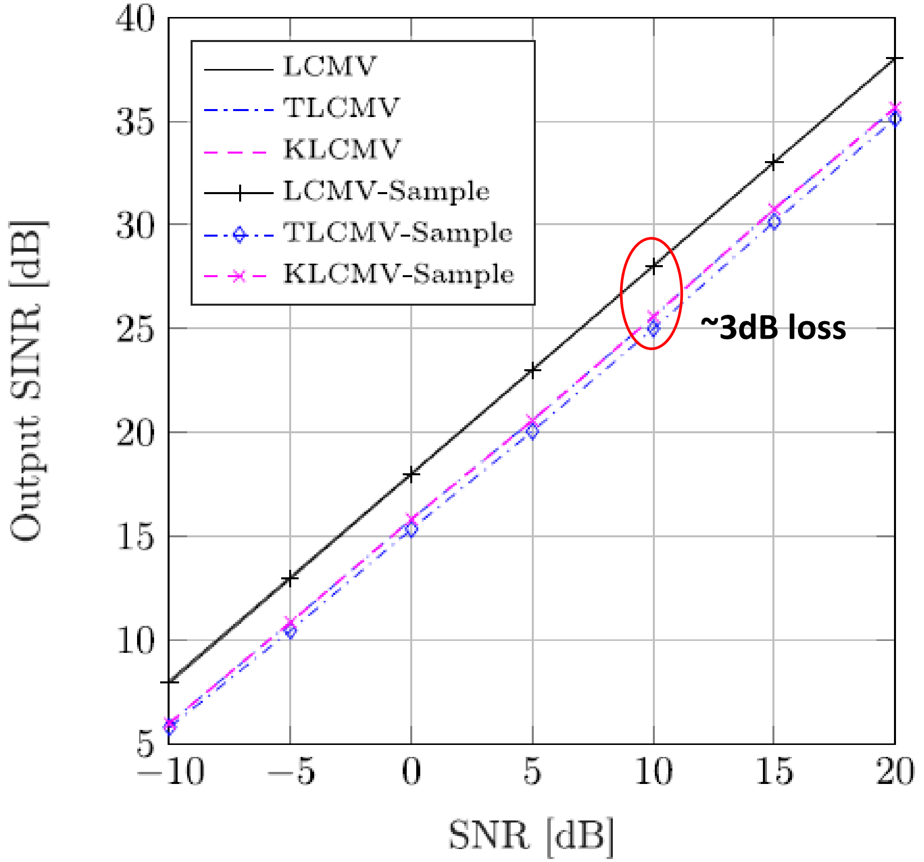
1. Random initialization for $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$
2. Optimize for $\boldsymbol{w}_1$ with $\boldsymbol{w}_2, \boldsymbol{w}_3$ fixed – $O(N_1^3)$ multiplications
3. Optimize for $\boldsymbol{w}_2$ with $\boldsymbol{w}_1, \boldsymbol{w}_3$ fixed – $O(N_2^3)$ multiplications
4. Optimize for $\boldsymbol{w}_3$ with $\boldsymbol{w}_1, \boldsymbol{w}_2$ fixed – $O(N_3^3)$ multiplications
5. Has converged? If not, go back to step 2

$O(N_1^3 + N_2^3 + N_3^3)$ vs. $O(N^3)$

Each filter is updated with alternating optimization methods

# Simulation Results - [Ribeiro et al'2019]




**Significant reduction of multiplications with small performance losses**

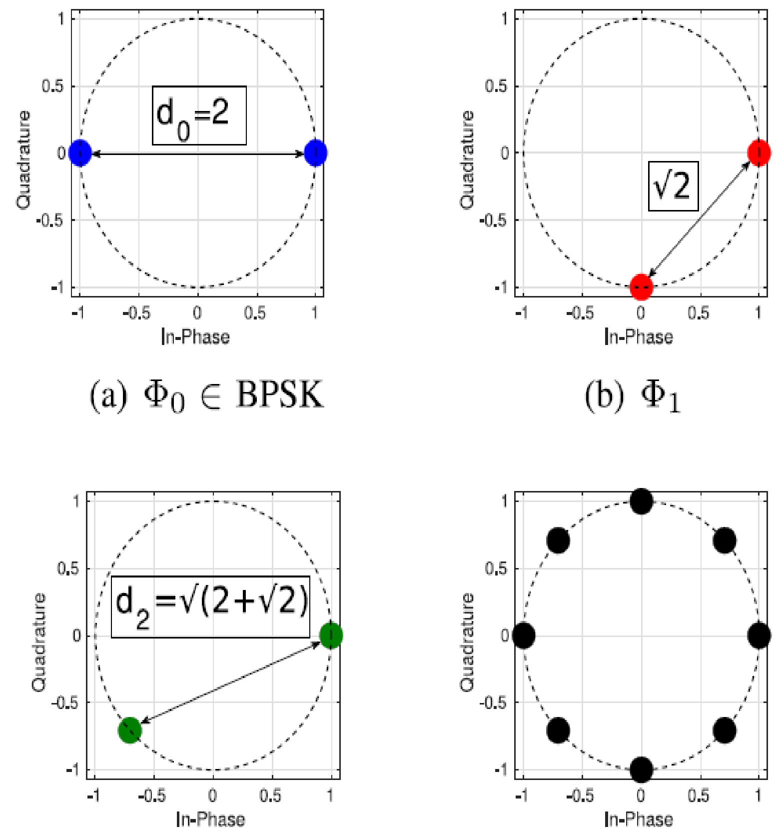
# Multi-linear constellation design

# Multi-linear constellation design

**Principle**

Any *M*-PSK constellation can be factorized into $P \leq \log_2 M$ different constellation sets:

$$\Phi = \Phi_0 \otimes \Phi_1 \cdots \otimes \Phi_{P-1}$$




(a) $\Phi_0 \in$ BPSK

(b) $\Phi_1$

(c) $\Phi_2$

(d) $\Phi = \Phi_0 \otimes \Phi_1 \otimes \Phi_2$

**Signal model**

$$\boldsymbol{y}[k] = h[k]\boldsymbol{x}[k] + \boldsymbol{n}[k]$$

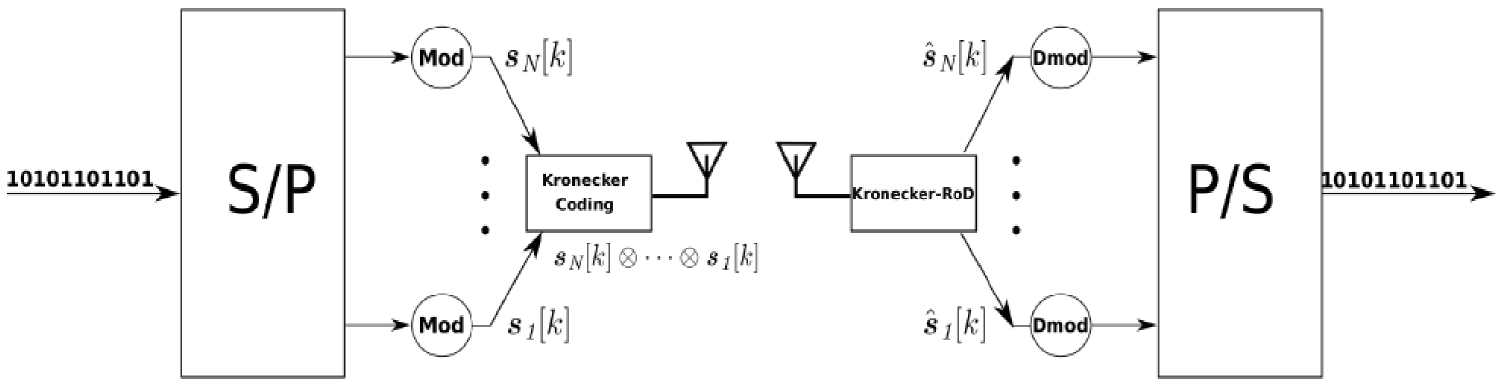
with

$$\boldsymbol{x}[k] = \boldsymbol{s}_N[k] \otimes \cdots \otimes \boldsymbol{s}_1[k]$$

Multi-linear *M*-PSK constellation

# Multi-linear constellation design

**Transceiver**




- Received signal after matched filtering (MF)

$$\hat{\boldsymbol{y}}[k] = h^*[k]\boldsymbol{y}[k]$$

- Decoding as *N*-th order rank-one tensor approx. problem

$$\min_{\boldsymbol{s}_1,\ldots,\boldsymbol{s}_N} \left\| \hat{\mathcal{Y}} - \boldsymbol{s}_1 \circ \cdots \circ \boldsymbol{s}_N \right\|_F^2$$

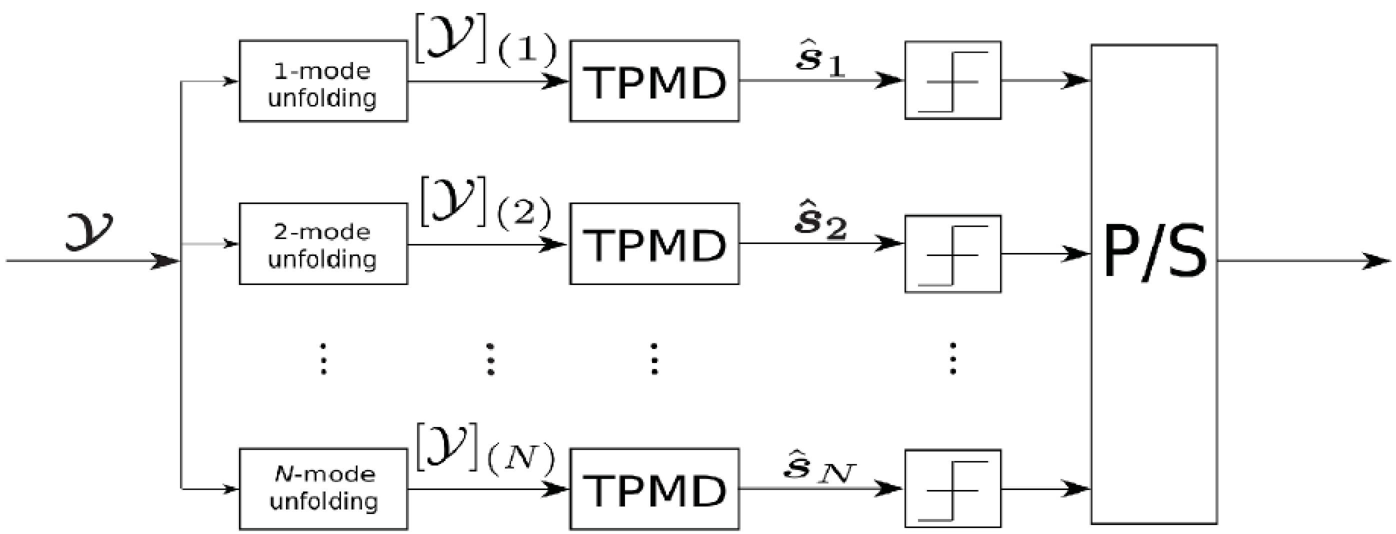
- Equivalent solution: maximize the tensor Rayleigh quotient

$$T(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_N) = \frac{\left|(\boldsymbol{s}_N \otimes \cdots \otimes \boldsymbol{s}_1)^T vec(\hat{\mathcal{Y}})\right|}{\|\boldsymbol{s}_1\|_2 \cdots \|\boldsymbol{s}_N\|_2}$$

# Multi-linear constellation design

**Receiver processing**

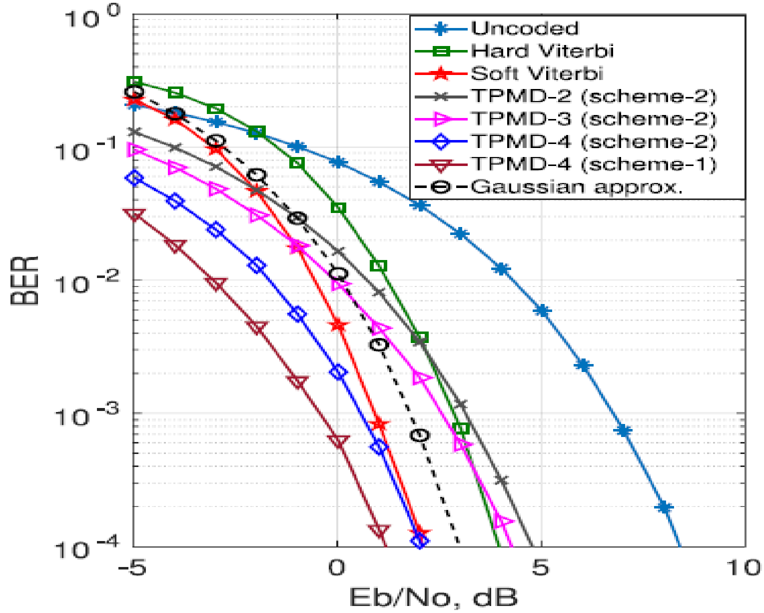
Kronecker Rank-One Detector (Kronecker-RoD)



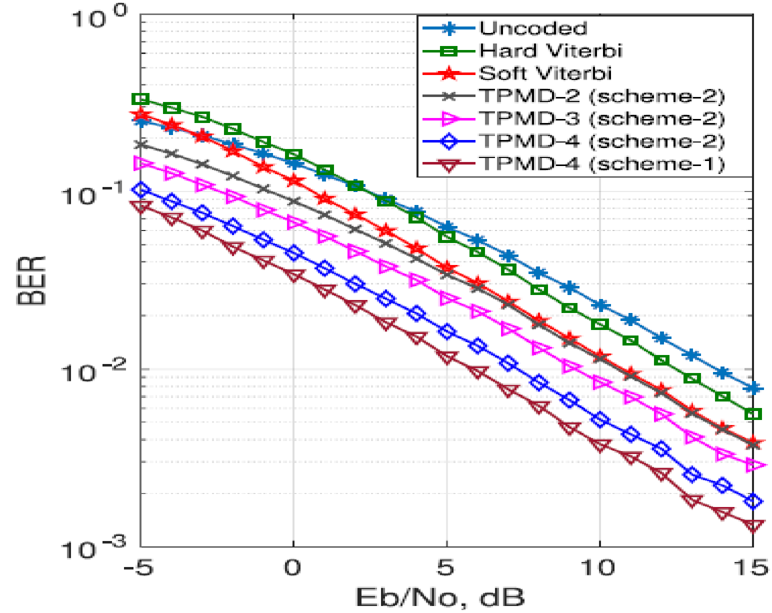

**Note:** Decoding can be parallelized → reduced latency

# Simulation Results

**Multi-linear QPSK modulation, ½ code rate**



(a) AWGN channel.



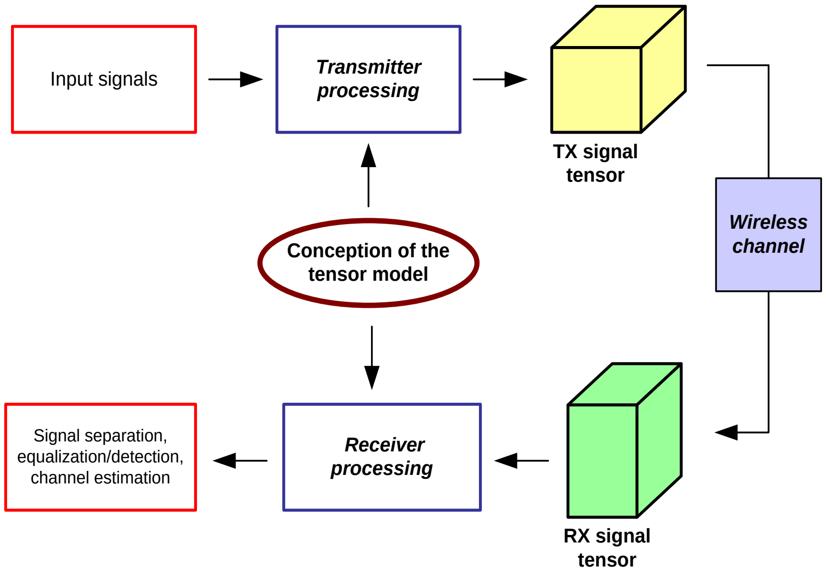
(b) Flat-fading channel.

- TPMD-4 outperforms hard/soft Viterbi decoding, especially at the low SNR
- Increasing the order of the tensor (→ number of Kronecker product terms) provides better results due to increased denoising capability

# Wrap up

- Tensors are powerful tools for modeling wireless communication systems (due to their multi-dimensional nature)
- Tensor modeling reveals “structured sparsity” and “structured low-rankness” of realistic wireless channels
- Joint (semi-)blind channel estimation & symbol detection, thanks to uniqueness property of tensor models
- Multi-linear filtering/beamforming schemes → significant complexity reduction with small performance losses
- Multi-linear constellations offer noise-robust detection; it can be exploited for phy-layer security

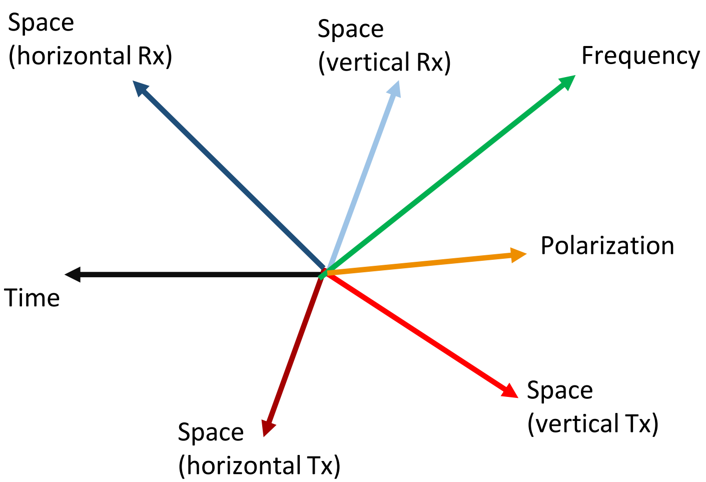
# Research directions & challenges




- Exploitation of block-sparsity, off-grid problems, multi-linear basis tracking solutions
- Optimize the design of tensor precoder/combiner and allocations to maximize performance
- Tensor-based algorithms that capture system "nonidealities" (e.g. unknown channel structure, hardware imperfections) → more realistic models & algorithms
- Distributed tensor-based algorithms (e.g. sensor networks/IoT, cell-free MIMO)

# Q&A

# Thank you!





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