Neural networks, flexible activation functions, and tensor approximations

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Overview

Flexible activations

Link to CPD

Coupled matrix/tensor factorization

Low-rank tensor recovery

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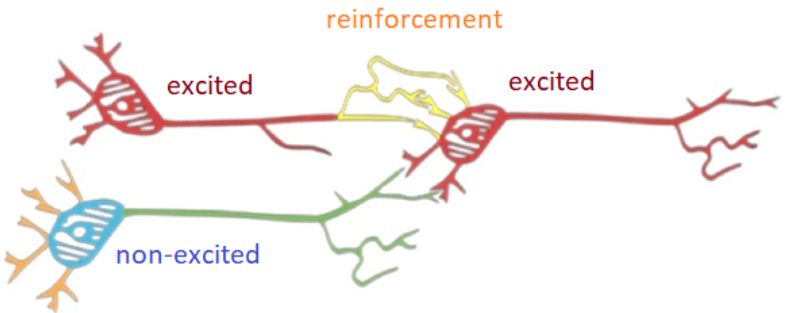
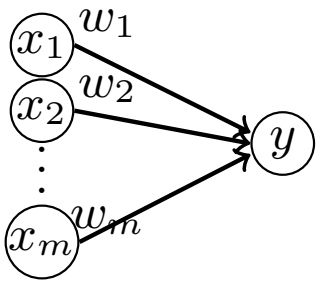
Feed-forward neural networks

- Huge success in computer vision [Krizhevsky et al., 2014]
- Very popular in data science, signal processing, engineering
- Basic problem supervised classification:
  Given a *training set* of objects $\mathbf{x}^{(k)} \in \mathbb{R}^m$ and (discrete) labels y_k,
  find ("learn") the map f such that $f(\mathbf{x}^{(k)}) \approx y_k$,
  which generalizes well: ($f(\mathbf{x}_{test}) \approx y_{test}$ on unseen *test* data)

Neural networks: early ideas



real neuron

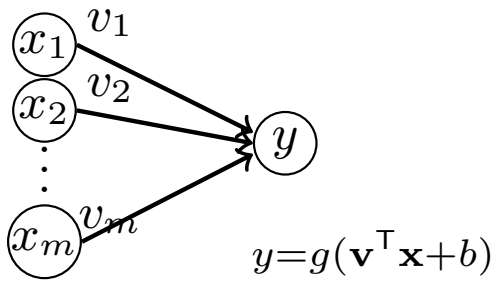


artificial neurons

$$y = \begin{cases} 1, & w_1 x_1 + \cdots + w_m x_m \geq \theta \quad \text{(a threshold)}, \\ 0, & \text{otherwise} \end{cases}$$

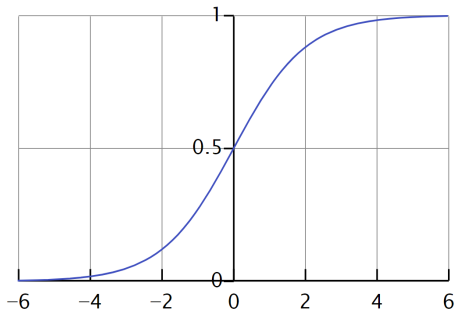
(McCulogh, Pitts, 1943), (Rosenblatt, 1958)

Perceptron and activation functions



- $\mathbf{v} \in \mathbb{R}^m$ — weights, b — "bias"
- $g(t)$ — activation ("thresholding") function

e.g., sigmoid



$$g(t) = \frac{1}{1+e^{-t}}$$

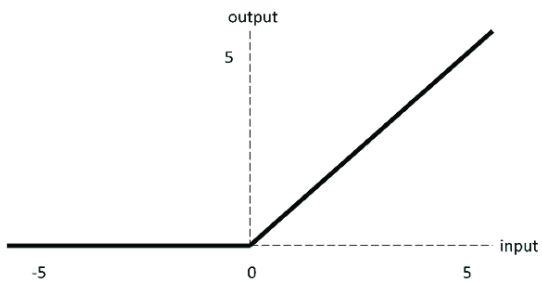
Main issues:

- level sets are hyperplanes
- classifier only for linearly separable classes

Modern feed-forward neural networks

Some key features:

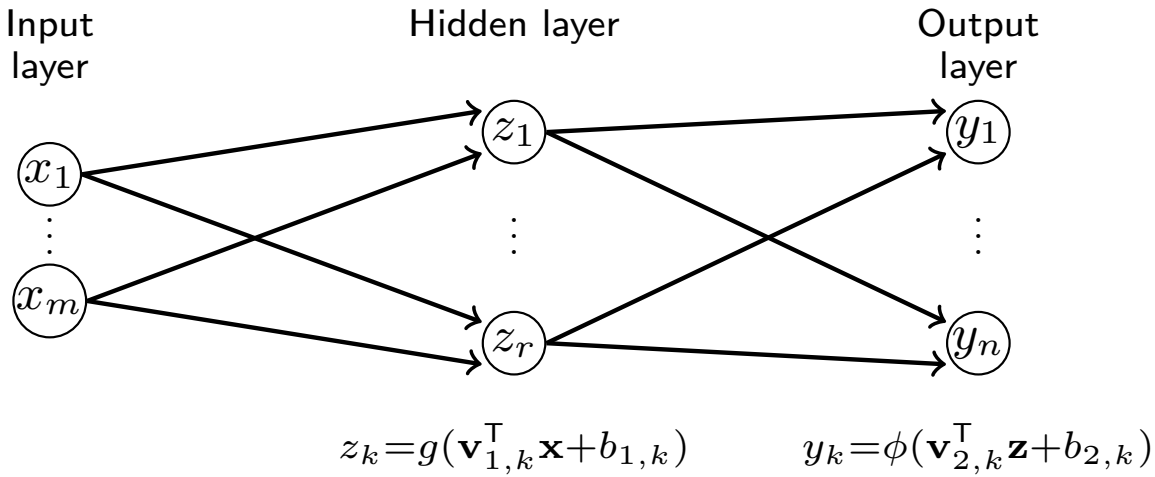
- Stacking several layers:
  lots of layers (deep learning) $\rightarrow$ better performance
- Shift to other activation functions, e.g. ReLu:



$$g(t) = \max(0, t)$$

- Structured weights (e.g., convolutional)
- Clever optimization algorithms, parallel computations on GPUs

Multilayer perceptron



- $\mathbf{v}_{1,k}$, $\mathbf{v}_{2,k}$ — vectors of weights, $b_{1,k}, b_{2,k}$ — biases
- g, ϕ — univariate activation functions

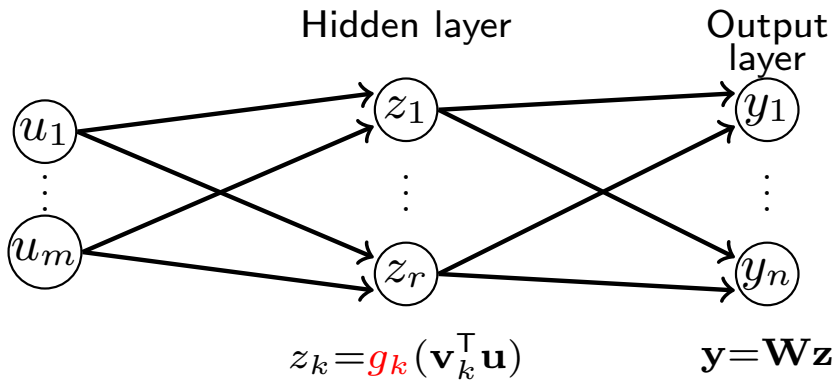
Theorem (Universal Approximation Theorem, e.g., (Pinkus, 1997))

Any continuous $\mathbf{f} : \Omega \to \mathbb{R}^n$ on compact Ω can be approximated to arbitrary accuracy by a multilayer perceptron $\mathbf{f}_\theta$ (with $\phi(t) = t$ and non-polynomial continuous g).

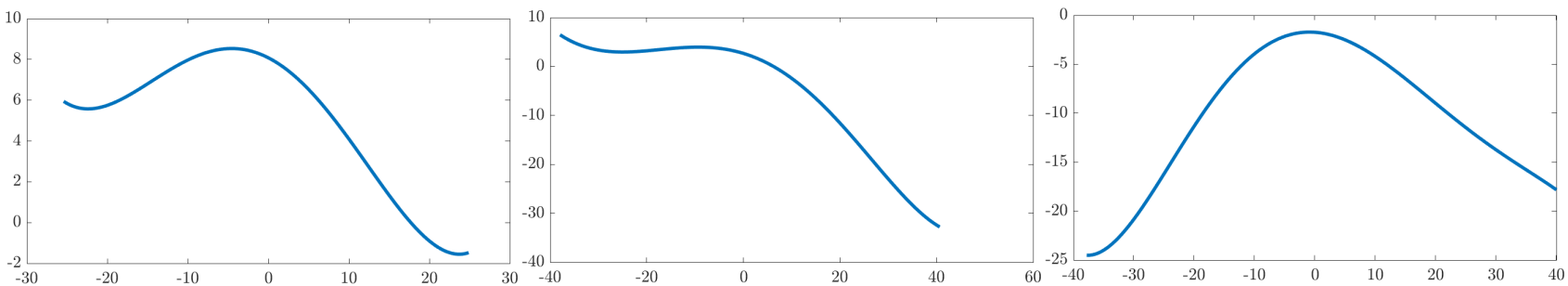
Can we add flexibility to g to achieve more compact representation?

From fixed to flexible activations

Simplest 2-layer model: no activation/bias at output



- fixed activations (classic setup): $g_k(t) = g(t + b_k)$
- flexible activations: different (ideally learned) g_k



Related work

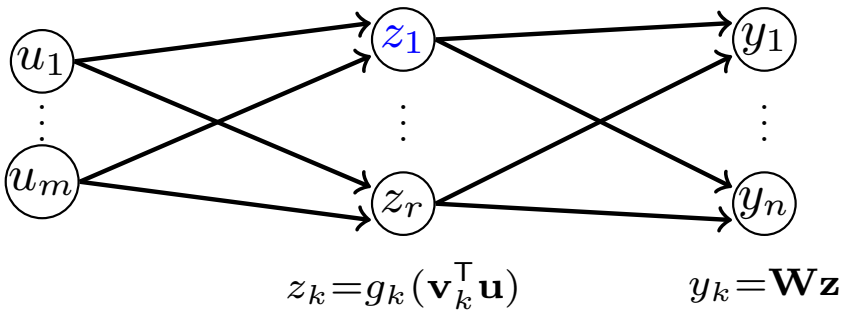
Neural network/approximation literature:

- deep spline networks [Aziznejad, Unser, 2019]
- ridge approximations [Lin, Pinkus, 1993]

Other work:

- independent component analysis [Comon, Jutten, 2010]
- block-structured nonlinear system identification [Dreesen et al., 2015]

Decomposition/approximation problem



Given a multivariate map $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^n$, decompose (approximate) it as

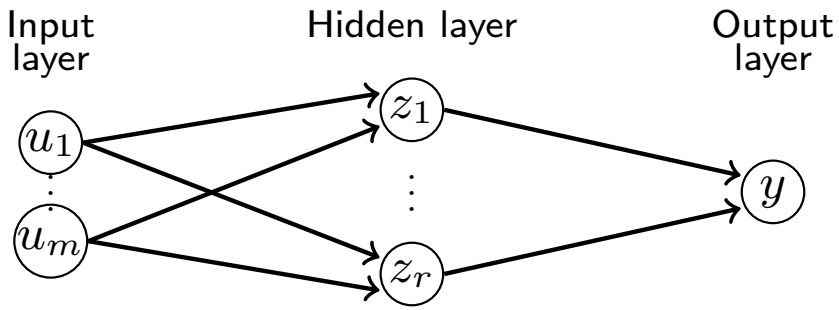
$$\mathbf{f}(\mathbf{u}) = \underbrace{\mathbf{w}_1 g_1(\mathbf{v}_1^\mathsf{T}\mathbf{u})}_{\text{single node (branch)}} + \cdots + \mathbf{w}_r g_r(\mathbf{v}_r^\mathsf{T}\mathbf{u}),$$

where

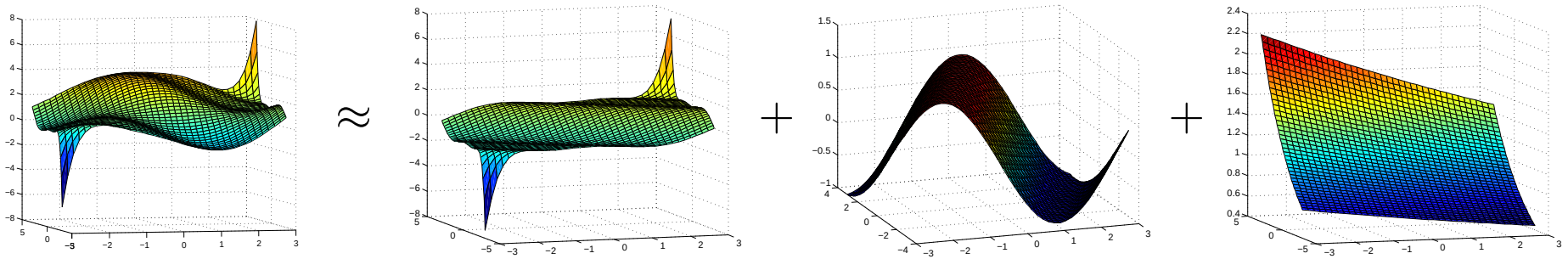
- $\mathbf{v}_k \in \mathbb{R}^m$, g_k — different univariate functions
- $\mathbf{W} = \begin{bmatrix}\mathbf{w}_1 & \cdots & \mathbf{w}_r\end{bmatrix} \in \mathbb{R}^{n\times r}$ — output weights

Geometric interpretation: ridge functions

Single output:



$$f(\mathbf{u}) = g_1(\mathbf{v}_1^\mathsf{T}\mathbf{u}) + \cdots + g_r(\mathbf{v}_r^\mathsf{T}\mathbf{u})$$

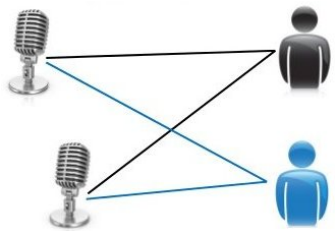


sum of "plane waves" or ridge functions *[Lin, Pinkus, 1993]*

Independent component analysis (source separation)

Linear mixing model:

$$\mathbf{x} = V\mathbf{s}$$



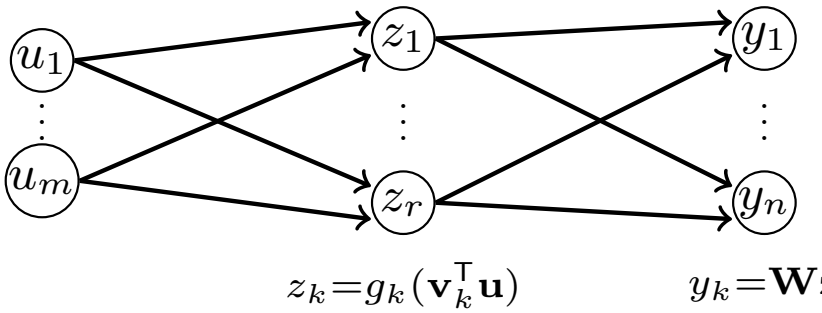
$$\underbrace{\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}}_{\text{sensors}} \in \mathbb{R}^n, \quad \underbrace{V = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_r \end{bmatrix}}_{\text{mixing matrix}}, \quad \underbrace{\mathbf{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_r \end{bmatrix}}_{\text{sources}} \in \mathbb{R}^r$$

if s_k are **independent**, the 2nd characteristic function has expansion

$$\Psi_{\mathbf{x}}(\mathbf{u}) = \psi_{s_1}(\mathbf{v}_1^{\mathsf{T}}\mathbf{u}) + \cdots + \psi_{s_r}(\mathbf{v}_r^{\mathsf{T}}\mathbf{u})$$

See [Comon, Jutten, 2010], [Rajih, Comon, 2006]

Compact matrix notation

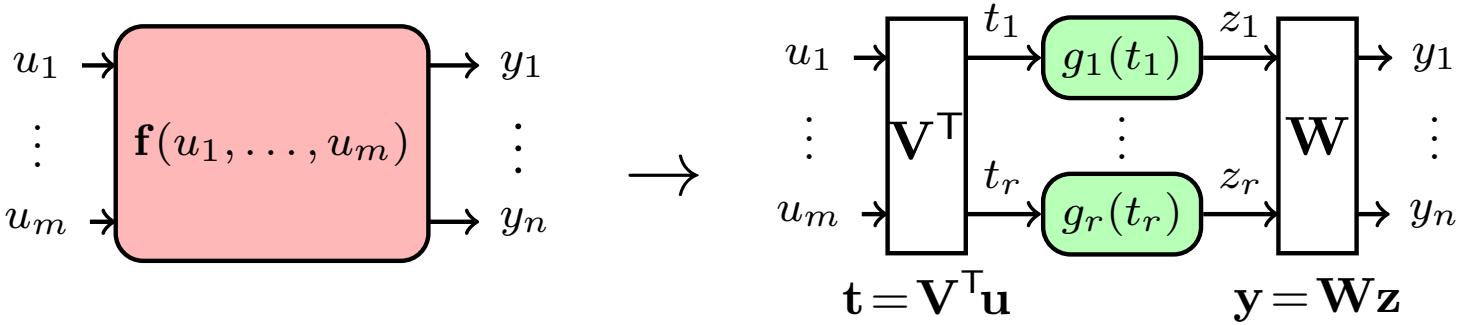


rewrite as $\boxed{\mathbf{f}(\mathbf{u}) = \mathbf{W}\mathbf{g}(\mathbf{V}^\mathsf{T}\mathbf{u})}$

with

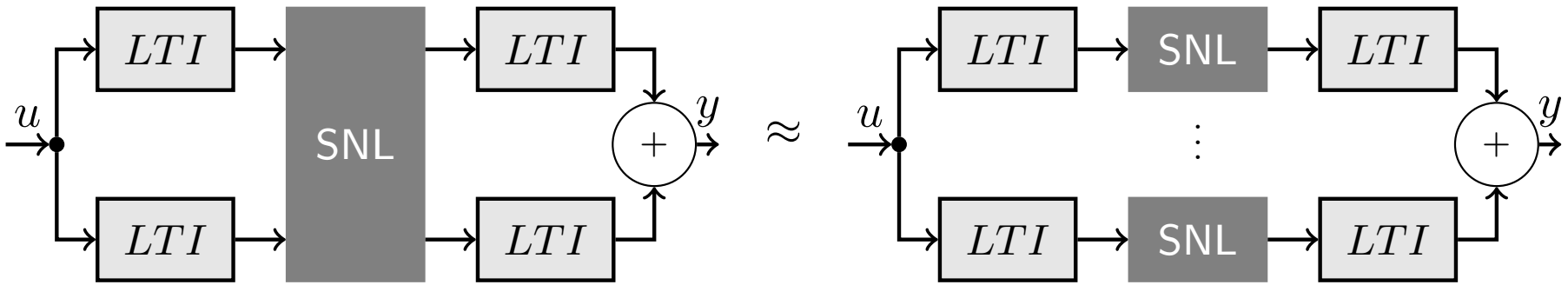
- $\mathbf{V} = \begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_r\end{bmatrix} \in \mathbb{R}^{m \times r}$
- $\mathbf{g}(t_1, \cdots, t_r) = [g_1(t_1) \cdots g_r(t_r)]^\mathsf{T}$

"Block-structured" form:

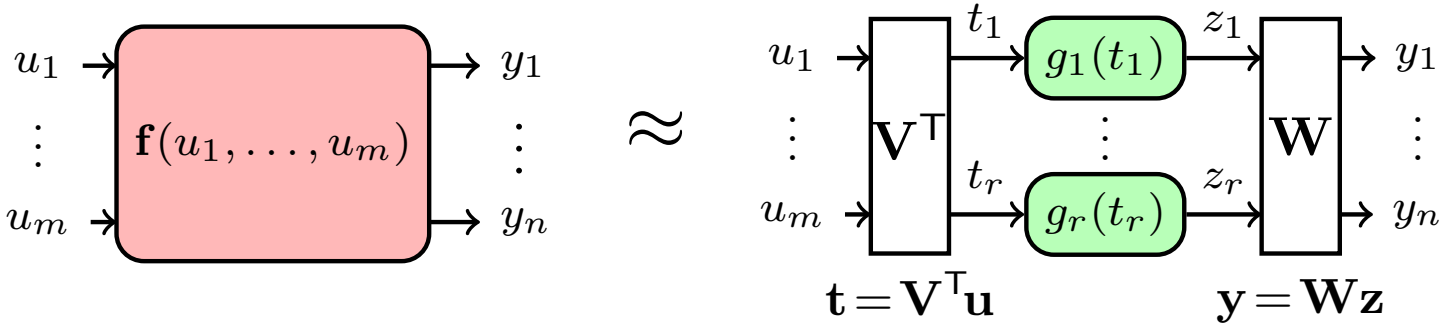


Nonlinear dynamical system identification

Identification of Parallel Wiener-Hammerstein systems



$\rightsquigarrow$ reduced to approximation of a multivariate map:



- called *decoupling* in system identification literature
- applicable to other block structures

Overview

Flexible activations

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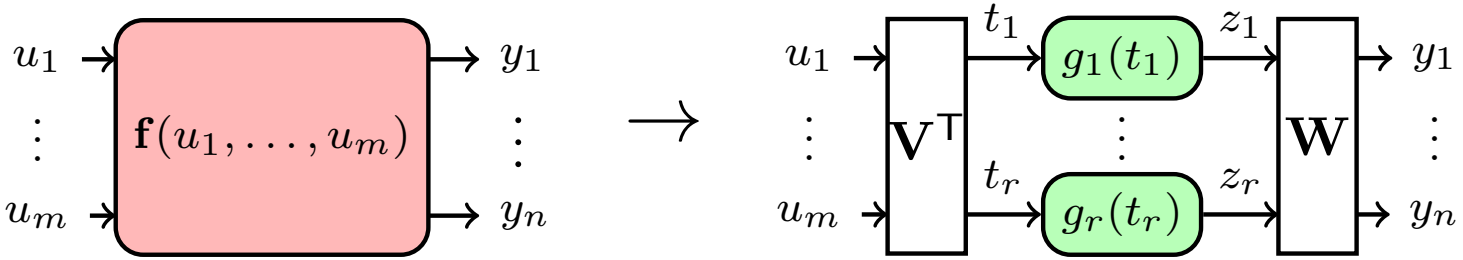
Exact factorization (decoupling)

Goal: Given $\mathbf{f}$, decompose it as $\mathbf{f}(\mathbf{u}) = \mathbf{W}\mathbf{g}(\mathbf{V}^\mathsf{T}\mathbf{u})$



with $\mathbf{V} \in \mathbb{R}^{m \times r}$, $\mathbf{W} \in \mathbb{R}^{n \times r}$, $\mathbf{g}(t_1, \cdots, t_r) = [g_1(t_1) \cdots g_r(t_r)]^\mathsf{T}$

Key idea of [Dreesen et al, 2015]: Jacobian of $\mathbf{f}$ has form

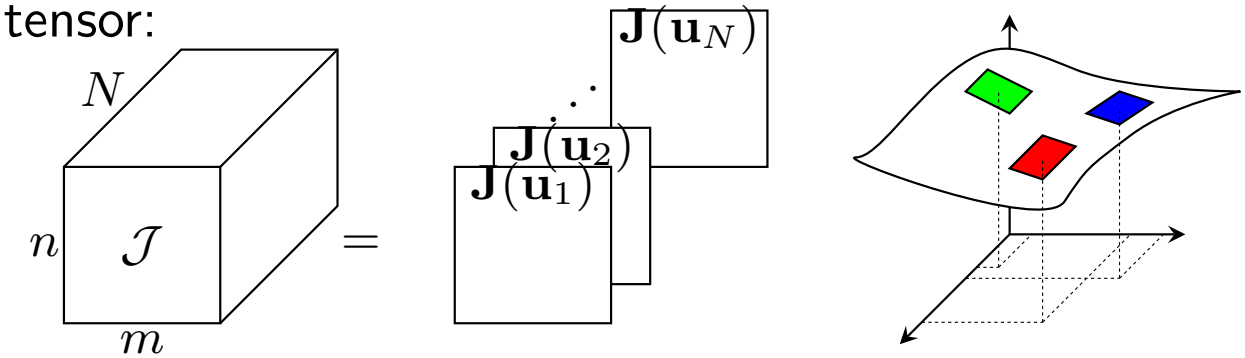
$$\boxed{\mathbf{J}_\mathbf{f}(\mathbf{u}) = \mathbf{W} \begin{bmatrix} g_1'(\mathbf{v}_1^\mathsf{T}\mathbf{u}) & & \\ & \ddots & \\ & & g_r'(\mathbf{v}_r^\mathsf{T}\mathbf{u}) \end{bmatrix} \mathbf{V}^\mathsf{T}}$$

only diagonal term depends on $\mathbf{u}$

Exact factorization: CPD of Jacobian tensor

Algorithm.

1. Evaluate $\mathbf{J_f}(\mathbf{u})$ at N "operating points" $\mathbf{u}_1, \ldots, \mathbf{u}_N \in \mathbb{R}^m$
2. Stack them into a tensor:



3. Joint matrix diagonalization $\leftrightarrow$ CPD $\boldsymbol{\mathcal{J}} = [\![\mathbf{W}, \mathbf{V}, \mathbf{H}]\!]$

$$\begin{matrix}\mathbf{J}(\mathbf{u}^{(1)}) = \mathbf{W}\mathbf{D}^{(1)}\mathbf{V}^{\mathsf{T}}, \\ \vdots \\ \mathbf{J}(\mathbf{u}^{(N)}) = \mathbf{W}\mathbf{D}^{(N)}\mathbf{V}^{\mathsf{T}}\end{matrix} \quad \longleftrightarrow \quad \mathcal{J} = \mathbf{w}_1 \circ \mathbf{v}_1 \circ \mathbf{h}_1 + \cdots + \mathbf{w}_r \circ \mathbf{v}_r \circ \mathbf{h}_r$$

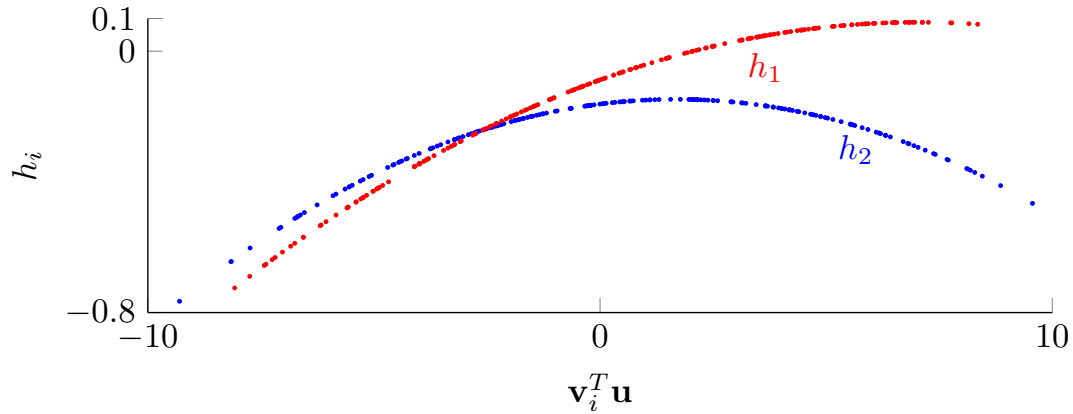
4. Retrieve $\mathbf{v}_k$, $\mathbf{w}_k$ from factors $\mathbf{W}, \mathbf{V}$ of the CPD
5. $\mathbf{h}_k$ contains evaluations of g_k'

Factorization and exact CPD

$$\mathbf{f}(\mathbf{u}) = \mathbf{W}\mathbf{g}(\mathbf{V}^\mathsf{T}\mathbf{u}) \Rightarrow \mathcal{J} = \mathbf{w}_1 \circ \mathbf{h}_1 \circ \mathbf{v}_1 + \cdots + \mathbf{w}_r \circ \mathbf{h}_r \circ \mathbf{v}_r$$

where $(\mathbf{h}_k)_i = g_k'(\mathbf{v}_k^\mathsf{T}\mathbf{u}_i)$

An example of recovery:
(exact decomposition,
$m = n = 2$, $r = 2$)



thanks to uniqueness of CPD of $\mathcal{J}$

Issues:

- CPD is a relaxation: structure of $\mathbf{H}$ is lost
- need extra step to estimate g_k

Polynomial case

Decompose polynomial map $\mathbf{f}\colon \mathbb{R}^m \to \mathbb{R}^n$ (degree d) as

$$\mathbf{f}(\mathbf{u}) = \mathbf{w}_1 g_1(\mathbf{v}_1^\mathsf{T}\mathbf{u}) + \cdots + \mathbf{w}_r g_r(\mathbf{v}_r^\mathsf{T}\mathbf{u}), \tag{1}$$

with $\mathbf{v}_k \in \mathbb{R}^m, \mathbf{w}_k \in \mathbb{R}^n$, $g_k(t) = c_{1,k}t + c_{2,k}t^2 + \cdots + c_{d,k}t^d$

[Comon, Qi, U., 2015], [U., Dreesen, Ishteva, 2020]: (1) is equivalent to a coupled tensor decomposition

$$\begin{aligned} \mathcal{T}^1 &= [\![\mathbf{W}, \mathbf{V}, \mathbf{c}_1^\mathsf{T}]\!], \\ \mathcal{T}^2 &= [\![\mathbf{W}, \mathbf{V}, \mathbf{V}, \mathbf{c}_2^\mathsf{T}]\!], \\ &\vdots \\ \mathcal{T}^d &= [\![\mathbf{W}, \underbrace{\mathbf{V}, \ldots, \mathbf{V}}_{d \text{ times}}, \mathbf{c}_d^\mathsf{T}]\!], \end{aligned}$$

Useful for studying uniqueness [Comon, Qi, U., 2017] (X-rank theory)

Generic uniqueness of the factorization

$$\mathbf{f}(\mathbf{u}) = \mathbf{w}_1 g_1(\mathbf{v}_1^{\mathsf{T}}\mathbf{u}) + \cdots + \mathbf{w}_r g_r(\mathbf{v}_r^{\mathsf{T}}\mathbf{u}),\ g_k(t) = c_{1,k}t + c_{2,k}t^2 + \cdots + c_{d,k}t^d$$

Theorem ([Comon, Qi, U., 2017], simplified)

Let s, $1 \leq s \leq d$, and r be such that

$$r \leq \min\left(\binom{m+s-1}{s}, C(m,n,d)\right) n.$$

Then for a general r-term $\mathbf{f}(\mathbf{u})$, we can recover uniquely (up to scaling ambiguities) $\mathbf{w}_k$, $\mathbf{v}_k$ and $c_{j,k}$, $j \geq s$

Examples:

- overall identifiability: $r \leq nm$
- nonlinear terms can be identifiable for $r \leq n\frac{m(m+1)}{2}$

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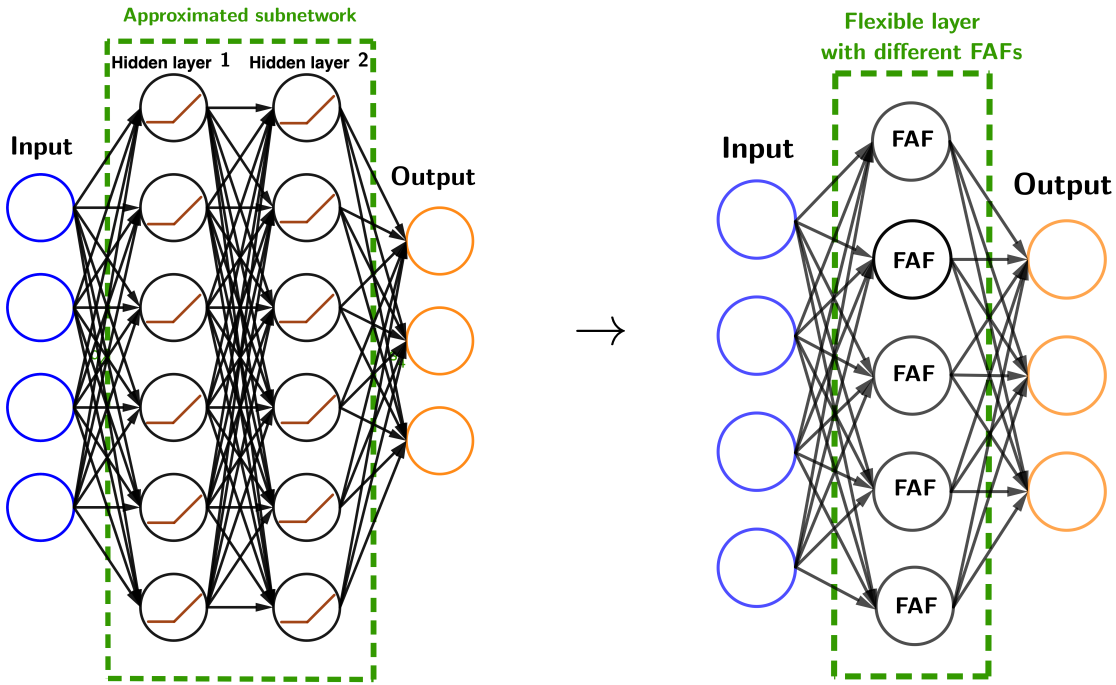
Tensor approximations for NN compression/learning

1. *(Lebedev et al., 2015)*
   Compress convolutional layer tensors ($\mathbf{W}$ is sparse and structured).
2. *(Novikov et al., 2016)*, Compression of fully-connected layers (e.g. tensorization of large matrix $\mathbf{W}$ +TT approximation)
3. *(Cohen et al., 2016)*: replace the nonlinear transfromation $g(\mathbf{W}^{\mathsf{T}}\mathbf{x} + \mathbf{b})$ with product pooling unit
   $\rightarrow$ (Khrulkov et al, 2018) link with tensor formats (TT, HT, ...)
4. *(Janzamin, et al. 2016)*: learning the linear layer $\mathbf{W}$ from tensor approximation of the score function

Framework: neural network compression

Goal: compress a part of a neural network:



with the flexible model: $\mathbf{f}(\mathbf{u}) \approx \mathbf{W}\mathbf{g}(\mathbf{V}^{\mathsf{T}}\mathbf{u})$

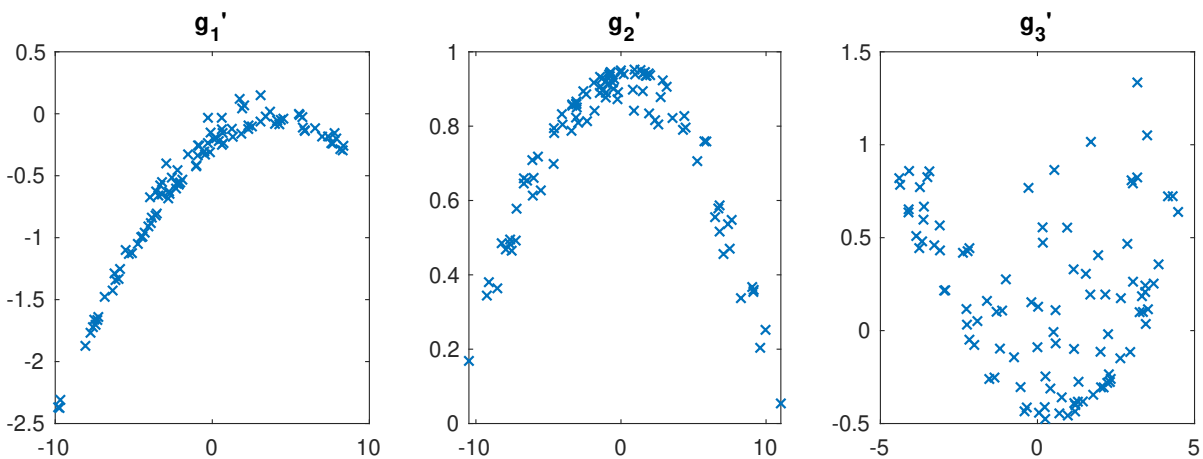
- pretrained network $\Rightarrow$ can evaluate derivatives
- use Jacobian tensor CP approximation

Drawbacks of the Jacobian approach

$$\min_{\mathbf{w}_k,\mathbf{v}_k,\mathbf{h}_k} \left\| \mathcal{J} - [\![\mathbf{W},\mathbf{V},\mathbf{H}]\!] \right\|_F, \quad \text{where} \quad \mathcal{J}_{:,:,i} = \mathbf{J}_{\mathbf{f}}(\mathbf{u}_i)$$

Issues:

- Need to estimate activation functions + loss of uniqueness

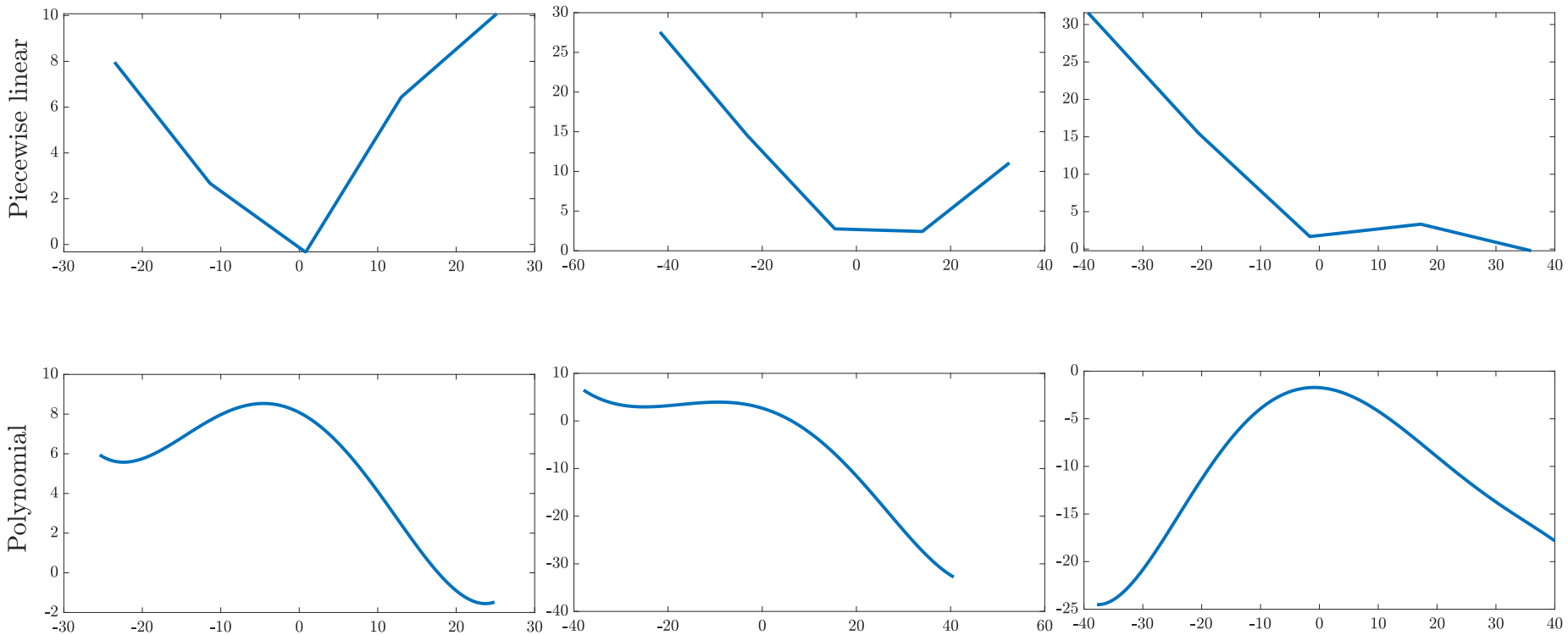


- derivatives are approximated instead of function values

Idea: add constraints

Activation function from a parameterized basis:

$$g_l(t) = c_{0,l} + c_{1,l}\phi_1(t) + \cdots + c_{d,l}\phi_d(t).$$



Proposed approach: coupled tensor-matrix factorization

For a base $g_l(t) = c_{0,l} + c_{1,l}\phi_1(t) + \cdots + c_{d,l}\phi_d(t)$, we solve

$$\min_{\mathbf{w}_k,\mathbf{v}_k,\mathbf{h}_k,\mathbf{z}_k} \underbrace{\left\|\mathcal{J} - [|\mathbf{W},\mathbf{V},\mathbf{H}|]\right\|^2}_{\text{Jacobian}} + \lambda \cdot \underbrace{\left\|\mathbf{F} - \mathbf{W}\mathbf{Z}^{\mathsf{T}}\right\|^2}_{\text{function evaluation (data fidelity)}}$$

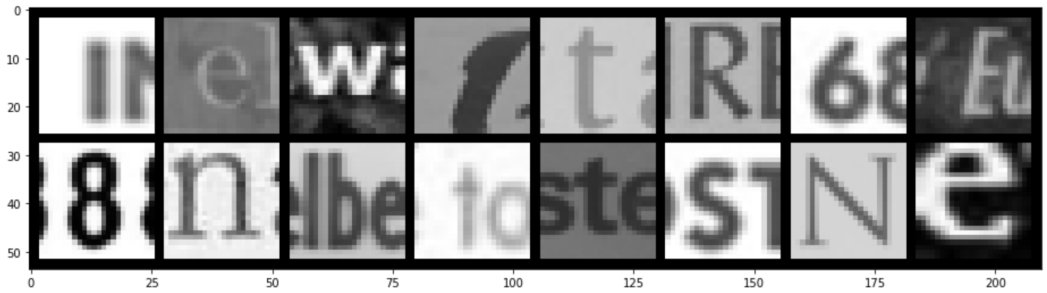
$$\text{subject to} \quad \mathbf{h}_l = \mathbf{X}_l \cdot \mathbf{c}_l, \quad \mathbf{z}_l = \mathbf{Y}_l \cdot \mathbf{c}_l, \quad \text{with}$$

$$\mathbf{X}_l = \begin{bmatrix} 0 & \phi_1'(\mathbf{v}_l^{\mathsf{T}}\mathbf{u}^{(1)}) & \cdots & \phi_d'(\mathbf{v}_l^{\mathsf{T}}\mathbf{u}^{(1)}) \\ \vdots & \vdots & & \vdots \\ 0 & \phi_1'(\mathbf{v}_l^{\mathsf{T}}\mathbf{u}^{(N)}) & \cdots & \phi_d'(\mathbf{v}_l^{\mathsf{T}}\mathbf{u}^{(N)}) \end{bmatrix}, \quad \mathbf{Y}_l = \begin{bmatrix} 1 & \phi_1(\mathbf{v}_l^{\mathsf{T}}\mathbf{u}^{(1)}) & \cdots & \phi_d(\mathbf{v}_l^{\mathsf{T}}\mathbf{u}^{(1)}) \\ \vdots & \vdots & & \vdots \\ 1 & \phi_1(\mathbf{v}_l^{\mathsf{T}}\mathbf{u}^{(N)}) & \cdots & \phi_d(\mathbf{v}_l^{\mathsf{T}}\mathbf{u}^{(N)}) \end{bmatrix}.$$

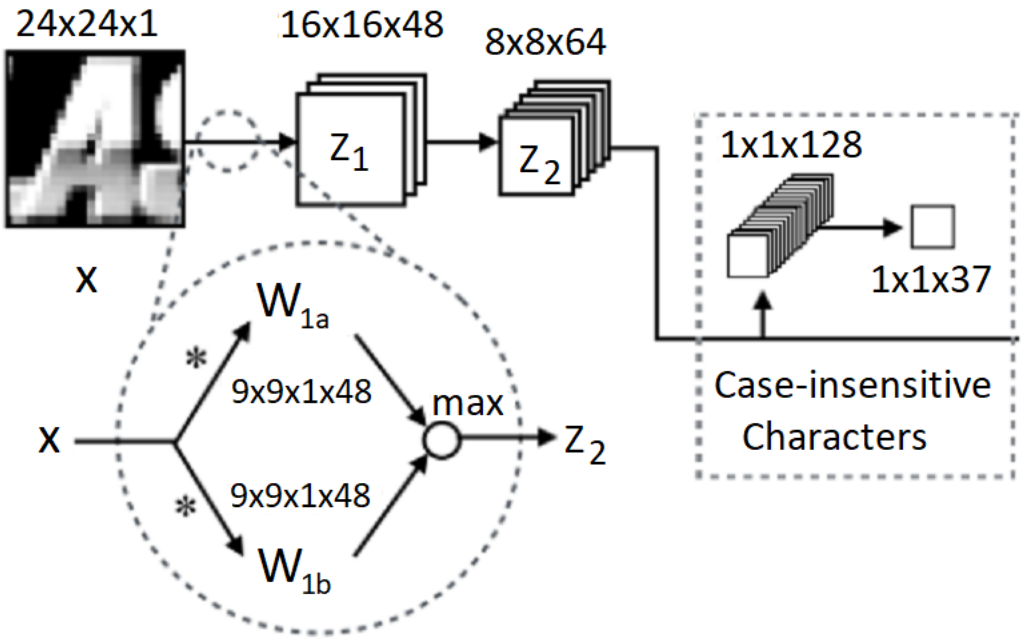
- constraints to preserve the structure
- data fidelity term to approximate well the predictions
  $\Rightarrow$ avoid fine-tuning

Test of compression

ICDAR 2003 dataset:
$\approx$163000 train, $\approx$ 5300 test,
36 classes



CharNet
[Jaderberg et al., 2014]



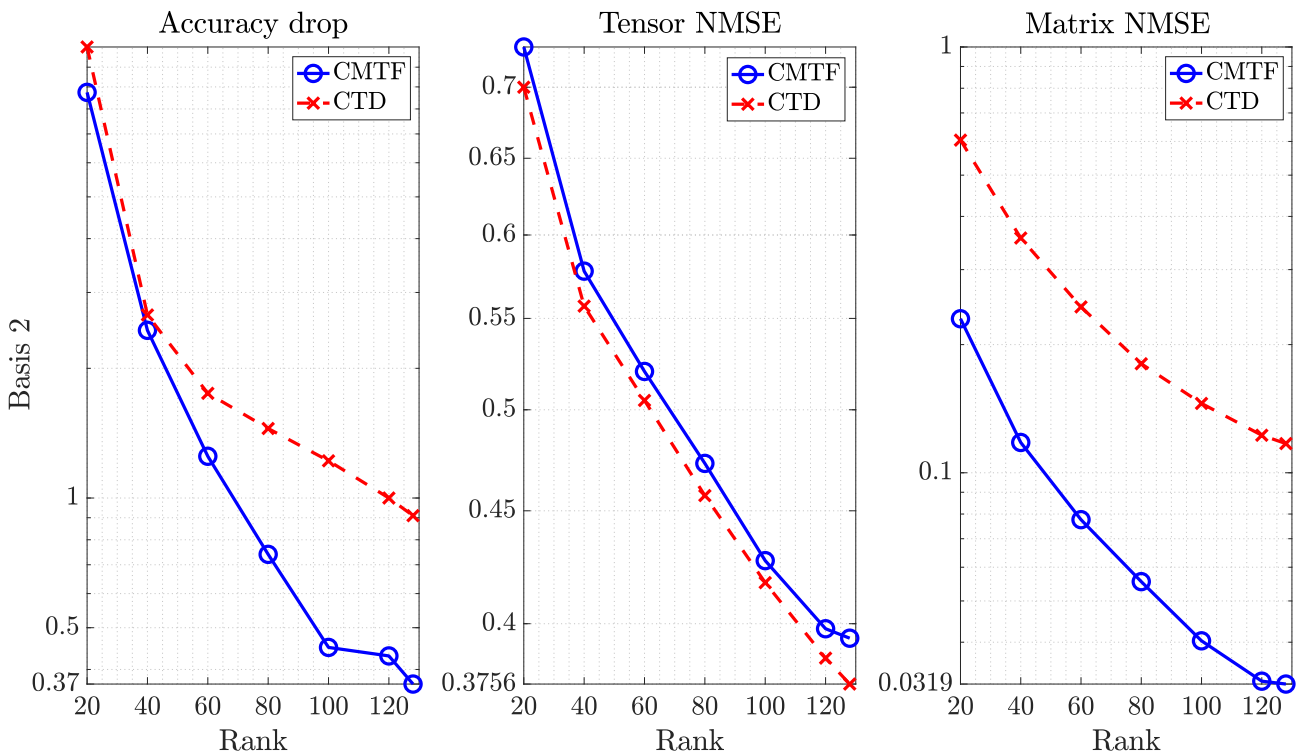
compress the conv3 layer (viewed as fully connected layer $\mathbb{R}^{4096} \to \mathbb{R}^{128}$)

CMTF compression performance

- use only 360 operating points $\mathbf{u}_i$ (10 per class from training set)
- no fine-tuning and 4x compression

$$\min_{\mathbf{w}_k,\mathbf{v}_k,\mathbf{h}_k,\mathbf{z}_k} \underbrace{\left\| \mathcal{J} - [|\mathbf{W},\mathbf{V},\mathbf{H}|] \right\|^2}_{\text{Jacobian}} + \lambda \cdot \underbrace{\left\| \mathbf{F} - \mathbf{W}\mathbf{Z}^{\mathsf{T}} \right\|^2}_{\text{function evaluation}}$$

subject to constraints on $\mathbf{H}$ and $\mathbf{Z}$



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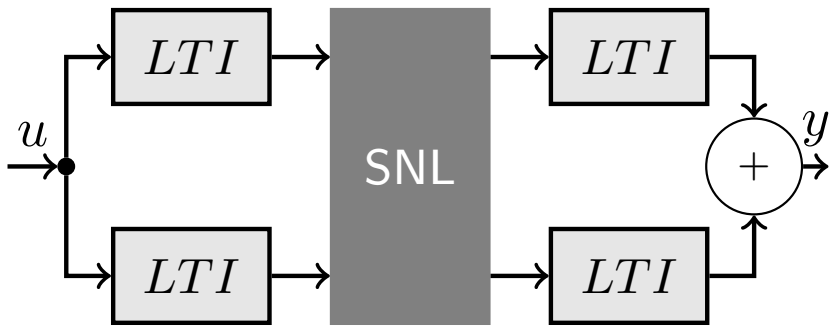
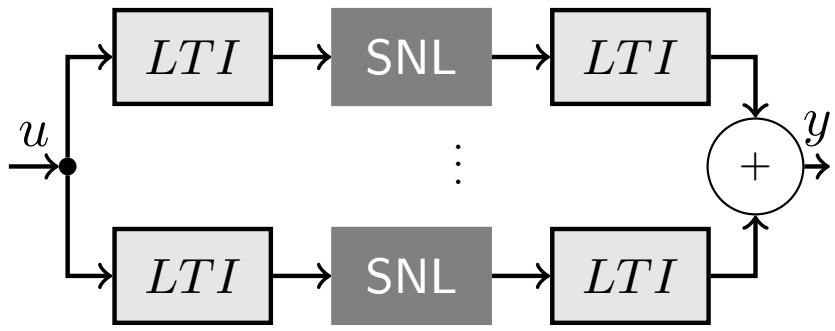
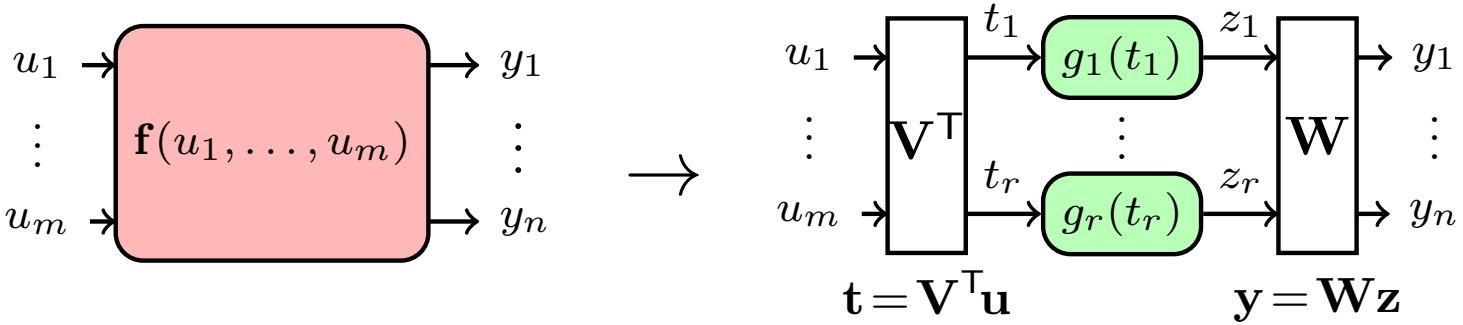




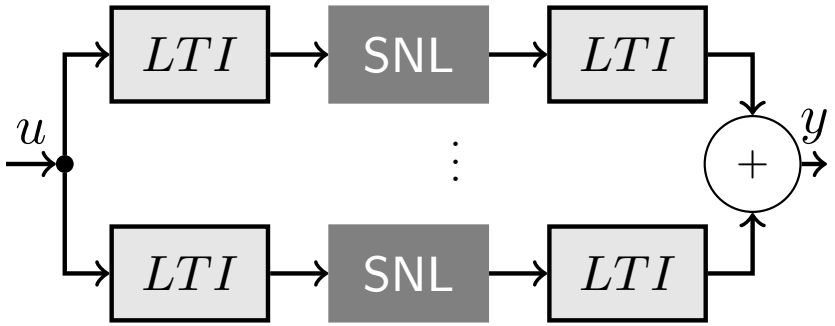
$\rightsquigarrow$ reduced to approximation of a multivariate map: $\mathbf{f}(\mathbf{u}) \approx \mathbf{W}\mathbf{g}(\mathbf{V}^{\mathsf{T}}\mathbf{u})$



- called *decoupling* in system identification literature
- applicable to other block structures

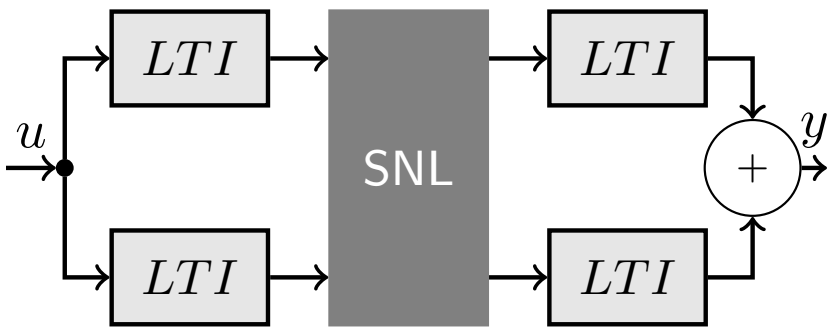
Parallel Wiener-Hammerstein system identification

Goal: estimate



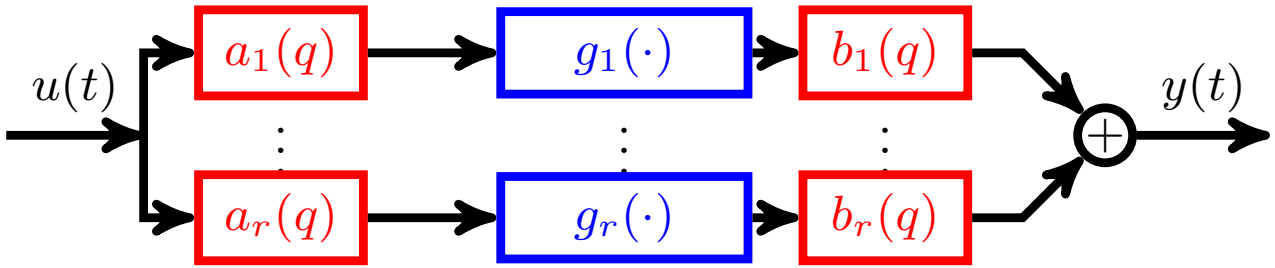
from input and output data $u(t)$, $v(t)$.

Current approaches are two-step: estimate first



Can we identify PWH system directly?

Parallel Wiener Hammerstein system: FIR case



Assumption:

- $a_k(q)$, $b_k(q)$ — finite impulse response (FIR) filters with lags L_1, L_2
- $g_k(\cdot)$ are univariate (static) functions

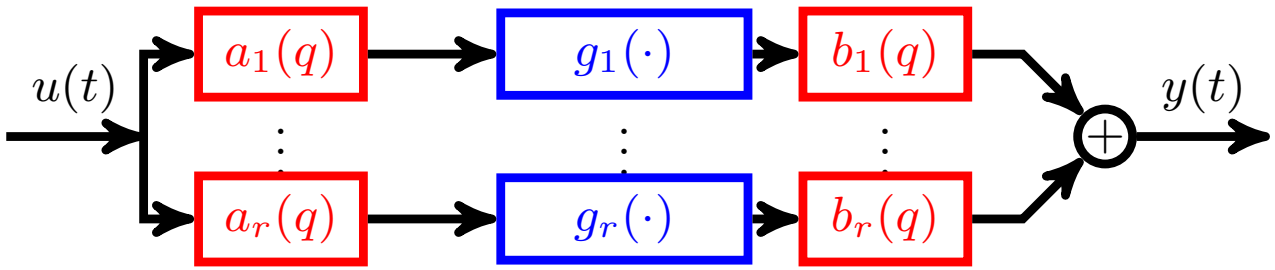
$$\mathbf{y} = \sum_{k=1}^{r} \mathbf{b}_k * (g_k(\mathbf{a}_k * \mathbf{u}))$$

where

- $\mathbf{a}_k \in \mathbb{R}^{L_1}$, $\mathbf{b}_k \in \mathbb{R}^{L_2}$ — filter coefficients
- $g_k(\cdot)$ applied elementwise to $\mathbf{a}_k * \mathbf{u}$ (filtered input)

convolutional NN with flexible activation functions

Our setup: FIR and polynomial activations



1. a_k, b_k are FIR with lags L_1 and L_2
   $\Rightarrow$ y depends on $L = L_1 + L_2 - 1$ past inputs:

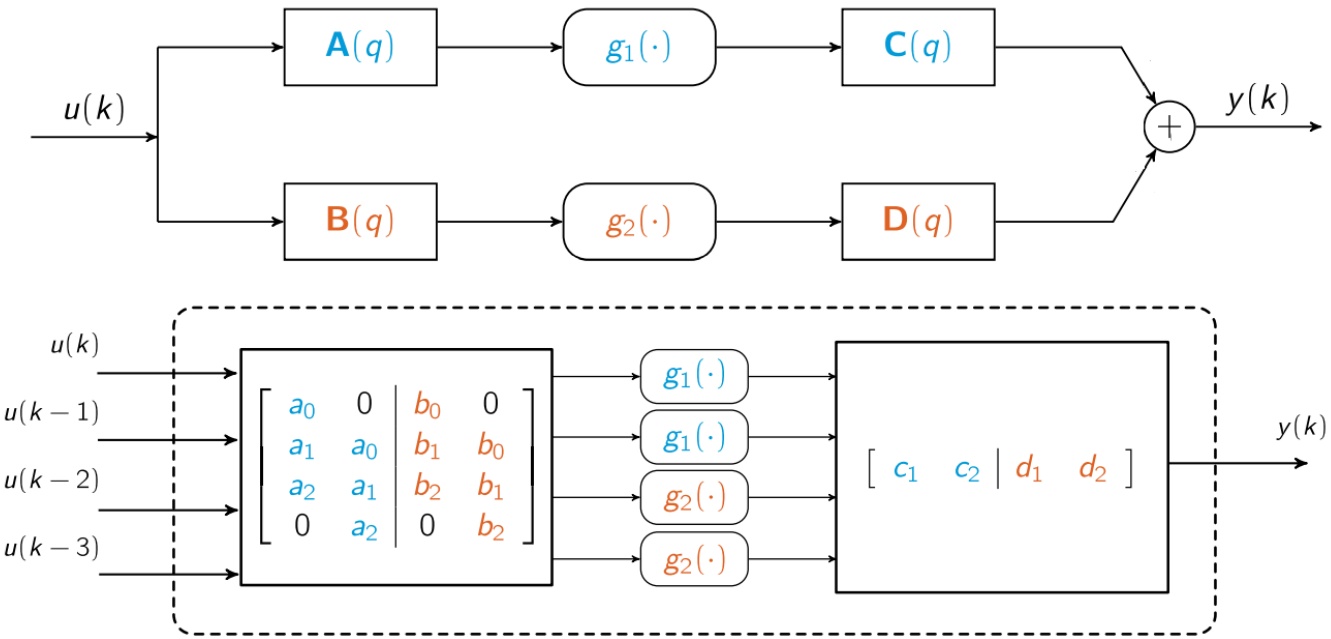
$$y(t) = f(u(t), u(t-1), \ldots, u(t-L+1)),$$

2. g_k — polynomial of degree d
   $\Rightarrow$ f is a multivariate polynomial in $(u(t), u(t-1), \ldots, u(t-L+1))$

   identification from truncated Volterra kernels up to order d

Apply plain decoupling to parallel Wiener-Hammerstein

[Dreesen, Ishteva, 2021]:



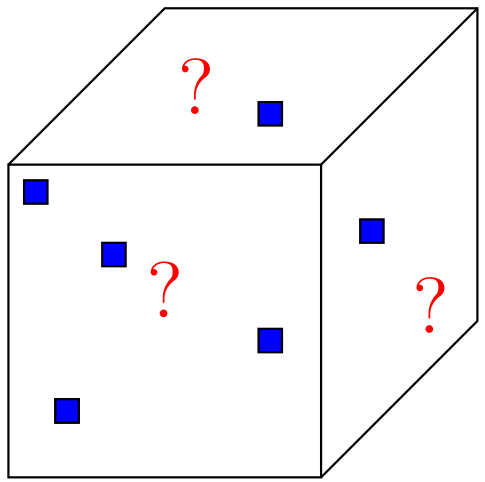
Drawbacks:

- Inflates tensor rank (rL_1 branches)
- Structured factors (similar to [Kibangou, Favier, 2010])

Our approach: use tensor recovery

Low-rank tensor recovery

- **Low-rank** tensor completion from few elements



$$= \quad [\![\mathbf{A}, \mathbf{B}, \mathbf{C}]\!]$$

- Tensor recovery: reconstruct a **low-rank** $\mathcal{T} = [\![\mathbf{A}, \mathbf{B}, \mathbf{C}]\!]$ from M samples (projections) $\mathscr{P}(\mathcal{T})$:

$$\mathscr{P}(\mathcal{T}) = \begin{bmatrix} \langle \mathcal{T}, \mathcal{S}_1 \rangle_F & \cdots & \langle \mathcal{T}, \mathcal{S}_M \rangle_F \end{bmatrix}^{\mathsf{T}}$$

— linear sampling (projection) operator

Key ideas of our approach

We mimick the CPD Jacobian method of [Dreesen et al. 2015], but

1. take operating points $\mu_k, k = 1, \ldots, N$ with Vandermonde structure:

$$\mathbf{u}_\mu = \begin{bmatrix} 1 & \mu & \mu^2 & \ldots & \mu^{L-1} \end{bmatrix}^\mathsf{T}, \quad \mu \in \mathbb{C}$$

2. split the nonlinearity into homogeneous parts:

$$f(\mathbf{u}) = f^{(0)} + f^{(1)}(\mathbf{u}) + \cdots + f^{(d)}(\mathbf{u})$$

3. where gradient of $f^{(s)}$ — partial contraction of s-th Volterra kernel:

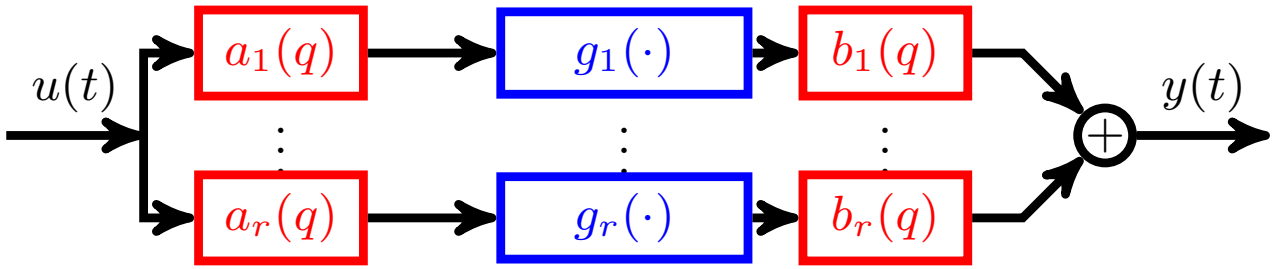
$$\nabla f^{(s)}(\mathbf{u}) = s \cdot \boldsymbol{\mathcal{H}}^{(s)} \bullet_2 \mathbf{u} \cdots \bullet_s \mathbf{u}.$$

4. stack the gradients into a "measurement" vector

$$\mathbf{z}_k = \begin{bmatrix} (\nabla f^{(1)}(\mathbf{u}_{\mu_k}))^\mathsf{T} & (\nabla f^{(2)}(\mathbf{u}_{\mu_k}))^\mathsf{T} & \cdots & (\nabla f^{(2)}(\mathbf{u}_{\mu_k}))^\mathsf{T} \end{bmatrix}$$

5. $\mathbf{z} = (\mathbf{z}_1, \ldots, \mathbf{z}_N)$ must be a sampling of a low-rank tensor

Main result



Proposition [U., Dreesen, Ishteva, 2021]
$\mathbf{z}$ is a projection of a rank-r polyadic decomposition

$$\mathbf{z} = \mathscr{P}(\boldsymbol{\mathcal{T}}), \quad \boldsymbol{\mathcal{T}} = \underbrace{\sum_{\ell=1}^{r} \mathbf{a}_\ell \otimes \mathbf{b}_\ell \otimes \mathbf{h}_\ell}_{[\![\mathbf{A},\mathbf{B},\mathbf{H}]\!]},$$

$\mathbf{a}_\ell \in \mathbb{R}^{L_1}$, $\mathbf{b}_\ell \in \mathbb{R}^{L_2}$, coefficients of g_ℓ can be recovered from $\mathbf{h}_\ell$

parallel Wiener-Hammerstein system — rank-r tensor recovery

Example

$r = 2$ branches, filter lengths $L_1 = 3$, $L_2 = 3$ with coefficients:

$$\mathbf{A} = \begin{bmatrix} 0.3 & 0.6 \\ -0.4 & 0.2 \\ 0.1 & 0.3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0.3 & 0.2 \\ 0.2 & 0.3 \\ 0.1 & 0.01 \end{bmatrix},$$

and nonlinearities $g_1(x_1) = 3x_1^3 - x_1^2 + 5, \quad g_2(x_2) = -5x_2^3 + 3x_2 - 7.$

Correct recovery with 30 random μ on the unit circle:

$$\widehat{\mathbf{A}} = \begin{bmatrix} 1 & 1 \\ -1.3333 - i0.8305 \cdot 10^{-8} & 0.3333 + i0.4203 \cdot 10^{-9} \\ 0.3333 - i0.2210 \cdot 10^{-8} & 0.4999 + i0.7007 \cdot 10^{-10} \end{bmatrix},$$

$$h_1(t) = t^2 - 0.2222t, \quad h_2(t) = t^2 - 0.2.$$

Tensor recovery

For $\mathbf{z} \in \mathbb{C}^M$, rank r, and sampling operator $\mathscr{P} : \mathbb{C}^{L_1 \times L_2 \times L_3} \to \mathbb{C}^M$ find

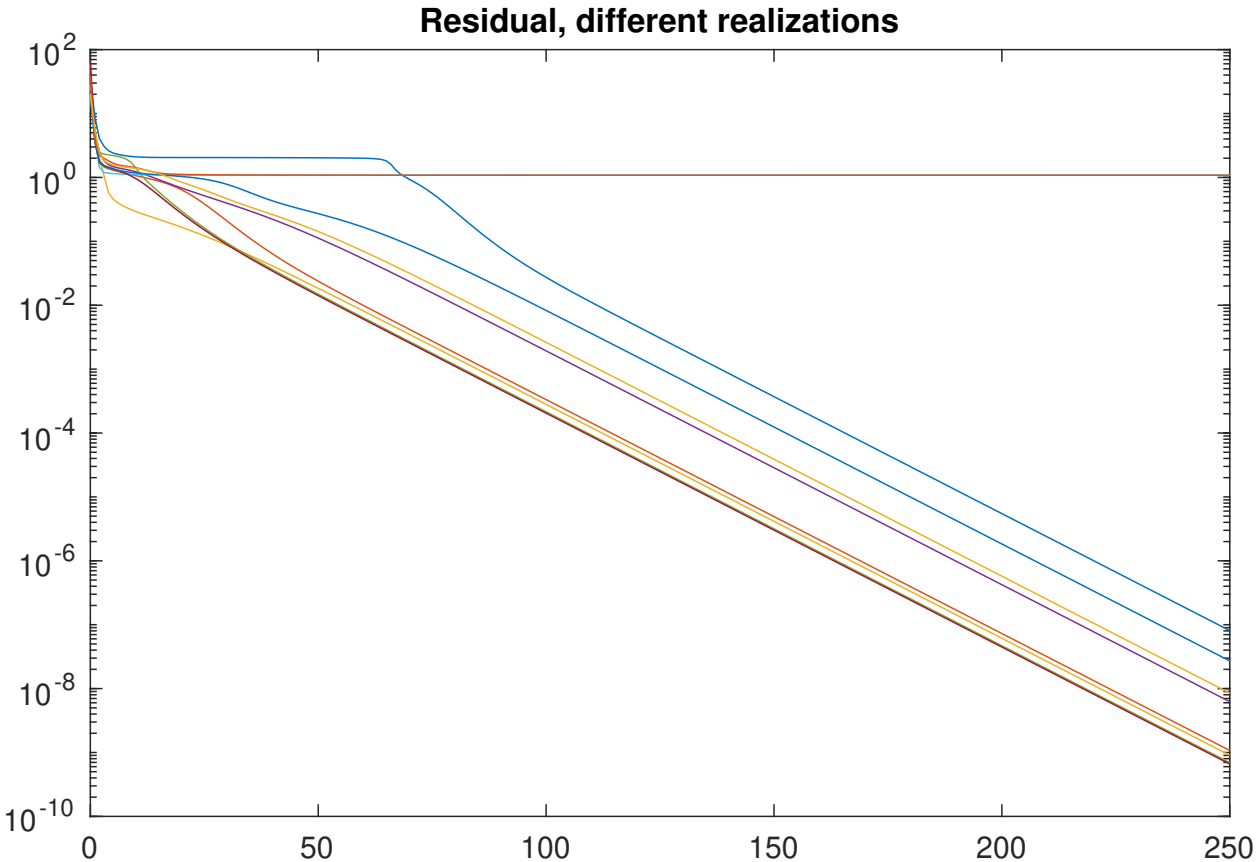
$$\min_{\mathbf{A} \in \mathbb{C}^{L_1 \times r}, \mathbf{B} \in \mathbb{C}^{L_2 \times r}, \mathbf{H} \in \mathbb{C}^{L_3 \times r}} \| \mathscr{P}(\llbracket \mathbf{A}, \mathbf{B}, \mathbf{H} \rrbracket) - \mathbf{z} \|_2^2,$$

Algorithm (alternating least squares).
Input: initializations $\mathbf{A}_0$, $\mathbf{B}_0$, $\mathbf{H}_0$.

1. For k=1,2,.... until a stopping criterion is satisfied
2. $\quad \mathbf{A}_k \leftarrow \arg\min_{\mathbf{A}} \| \mathscr{P}(\llbracket \mathbf{A}, \mathbf{B}_{k-1}, \mathbf{H}_{k-1} \rrbracket) - \mathbf{z} \|_2^2$;
3. $\quad \mathbf{B}_k \leftarrow \arg\min_{\mathbf{B}} \| \mathscr{P}(\llbracket \mathbf{A}_k, \mathbf{B}, \mathbf{H}_{k-1} \rrbracket) - \mathbf{z} \|_2^2$;
4. $\quad \mathbf{H}_k \leftarrow \arg\min_{\mathbf{H}} \| \mathscr{P}(\llbracket \mathbf{A}_k, \mathbf{B}_k, \mathbf{H} \rrbracket) - \mathbf{z} \|_2^2$.
5. End for

Dependence on intialization

Convergence for 10 initializations (prev. example):



Open questions and references

Ongoing work:

- deeper architectures with flexible activation functions: identifiability and algorithms
- optimization (better/scalable algorithms, convergence)
- robustness to perturbations, stability (link with uniqueness)
- link with other tensor approximation methods for NN compression

Thank you!

And all my collaborators:

- Jacobians (CPD-based methods):
  K. U., P. Dreesen and M. Ishteva., *Decoupling multivariate polynomials: interconnections between tensorizations*, JCAM, 2020.
- Generic uniqueness (X-rank):
  P. Comon, Y. Qi and K. U., *X-rank and identifiability for a polynomial decomposition model*, SIAGA, 2017.
- NN compression/ CMTF:
  Y.Zniyed, K. U., S. Miron, D. Brie, *Tensor-based framework for training flexible neural networks*, 2021, arXiv:2106.13542 .
- Tensor recovery for convolutional case:
  K. U., P. Dreesen, M. Ishteva, *Low-rank tensor recovery for Jacobian-based Volterra identificationof parallel Wiener-Hammerstein systems*, proceedings of SYSID 2021, arXiv:2109.09584 .