Tropical linear regression and low-rank approximation — a first step in tropical data analysis

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Workshop on Tensor Theory and Methods

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Outline

Overview

The Tropical Semiring

- The tropical semiring is $(\mathbb{R}_{\max}, \oplus, \odot)$, with :

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- $(\mathbb{R}_{\max}, \oplus, \odot) \simeq (\mathbb{R} \cup \{+\infty\}, \min, +)$

Applications

Tropical data arise naturally:

- Control theory   Game theory   Economics   ...

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A baby example:

- Given a direct-distances matrix $A = (a_{ij})$, find the longest distances.
- Consider $A^2 = A \odot A$, i.e.,

$$(A^2)_{ij} = \max_k (a_{ik} + a_{kj}),$$

which gives the longest path between i and j of length 2.

- $A + A^2 + \cdots$ gives longest paths of all lengths.

ReLU NN

Another example [1]:

$$\left\{ \begin{array}{l} \text{feedforward ReLU neural networks} \\ \text{with integer weights} \end{array} \right\} \simeq \{\text{tropical rational maps}\}$$

[1] L. Zhang, G. Naitzat, L.-H. Lim, *Tropical geometry of deep neural networks*, ICML, 2018

Ground states of spin glasses [2]

Finding the ground sate of the Ising spin glass with the energy function

$$E(\{\sigma\}) = -\sum_{i<j} J_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i,$$

where $\{\sigma\} \in \{\pm 1\}^N$ a configuration of N Ising spins.
For the partition function $Z = \sum_\sigma e^{-\beta E}$, the ground state energy

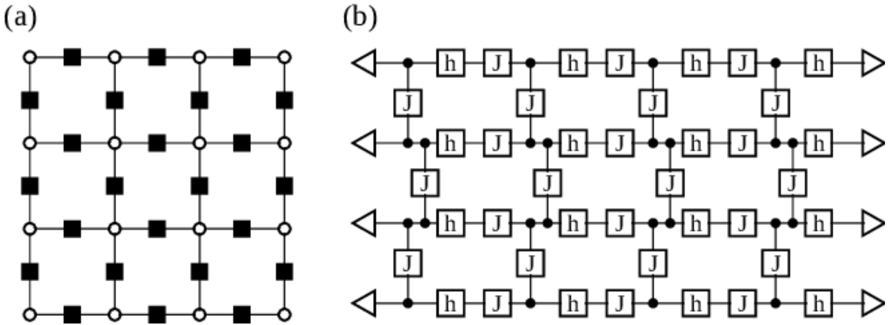
$$E^* = -\lim_{\beta\to\infty}(1/\beta)\ln Z = -\lim_{\beta\to\infty}(1/\beta)\ln \sum_\sigma \prod_{i<j} e^{\beta J_{ij}\sigma_i\sigma_j} \prod_i e^{\beta h_i \sigma_i}$$

Taking the zero-temperature limit,

$$\lim_{\beta\to\infty}(1/\beta)\ln(e^{\beta x} + e^{\beta y}) = x \oplus y, \quad (1/\beta)\ln(e^{\beta x} \times e^{\beta y}) = x \odot y.$$

[2] J. Liu, L. Wang, P. Zhang, *Tropical tensor network for ground states of spin glasses*, Physical Review Letters, 2021

Tropical tensor networks



Example 1: function approximation

- X: compact metric
- Y: nonempty
- $\mathcal{C}(X)$: the space of continuous functions on X
- $\mathcal{B}(Y)$: the space of bounded functions on Y.
- Given $V : X \times Y \to \mathbb{R}$ bounded, find f and g such that

$$\inf_{f \in \mathcal{C}(X),\, g \in \mathcal{B}(Y)} \sup_{x \in X, y \in Y} |V(x, y) - f(x) - g(y)|$$

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will see later : $f + g$ is a best rank-one approximation of V

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- For invitation $j \in [q]$, company $i \in [n]$ asks for the price p_{ij} (public)
- Secret evaluation $0 < f_i \leqslant 1$ (**technical quality**)
- The decision maker minimizes his expected cost:

$$\min_{i \in [n]} p_{ij} f_i^{-1}$$

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will see later: $\forall j \in [q]$, $V_{\cdot j}$ lies in some hyperplane H_a

Tropical data analysis

Given tropical data, we would like to do

- Tropical linear regression:
  Given m points, find a hyperplane which best fits these data.
- Tropical low-rank approximation:
  Given a tropical matrix, find a best low-rank approximation.

Overview

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- $C \subseteq (\mathbb{R}_{\max})^n$ is a **tropical (convex) cone** if $\forall x, y \in C, \forall \lambda \in \mathbb{R}_{\max}$:

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- For a set $\mathcal{V} \subseteq (\mathbb{R}_{\max})^n$, the tropical cone generated by $\mathcal{V}$:

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- The **tropical projective space** $\mathbb{P}(\mathbb{R}_{\max})^n$ is the quotient

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- Given $A, B \subseteq \mathbb{P}(\mathbb{R}_{\max})^n$, the **one-sided Hausdorff distance**:
$$\mathrm{d}_H(A, B) := \sup_{a \in A} \inf_{b \in B} \|a - b\|_H \ .$$

Tropical hyperplanes

Given $a \in \mathbb{P}(\mathbb{R}_{\max})^n$, we define the **tropical hyperplane**:

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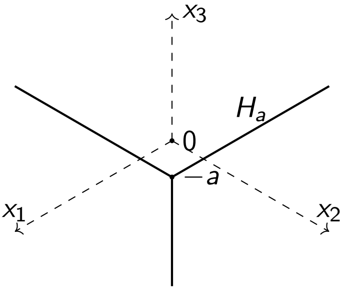

Figure: The hyperplane H_a with $a = (0,0,1)^\top$ and in $\mathbb{P}(\mathbb{R}_{\max})^3$.

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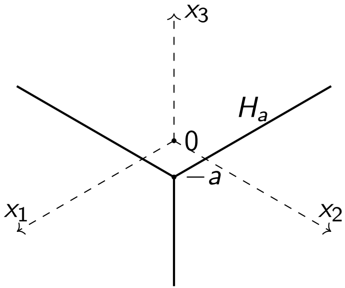

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Given a set $\mathcal{V} \subseteq \mathbb{R}^n_{\max}$, we solve **the tropical linear regression problem**:

$$\inf_{a \in \mathbb{P}(\mathbb{R}_{\max})^n} \mathrm{d}_H(\mathcal{V}, H_a) \ .$$

Inner radius

For a subset $\mathcal{V}$ of $(\mathbb{R}_{\max})^n$, the *inner radius* of $\mathcal{V}$ is:

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Theorem (Akian–Gaubert–Q.–Saadi)

$$\inf_{a \in \mathbb{P}(\mathbb{R}_{\max})^n} \mathrm{d}_H(\mathcal{V}, H_a) = \text{in-rad}(\mathcal{V}).$$

Question: how to compute in-rad$(\mathcal{V})$?

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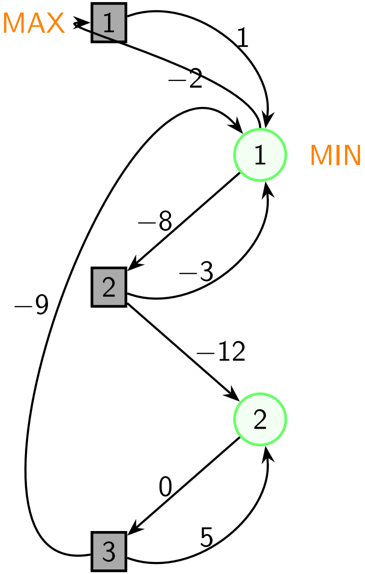
- $-V_{ik} + V_{jk}$ is the instantaneous payment made by Min to Max
- *Asymetry: Min can play tit for tat but Max cannot!*

Baby example



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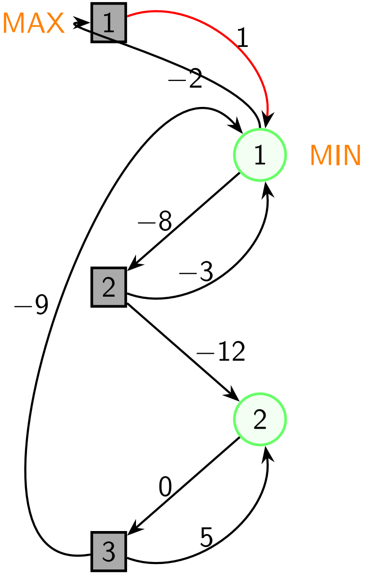

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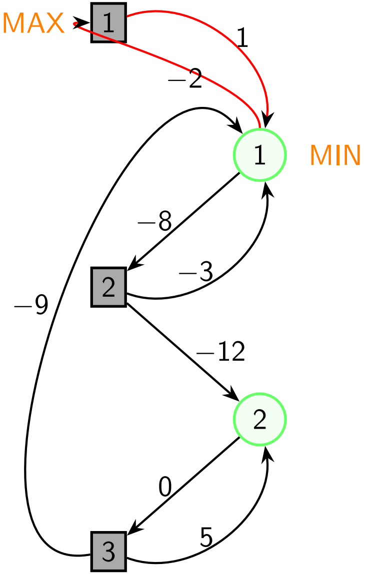

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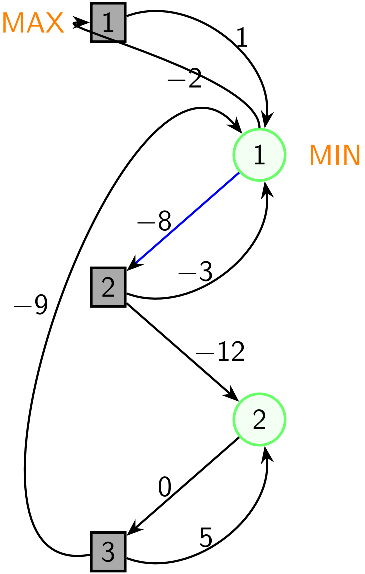

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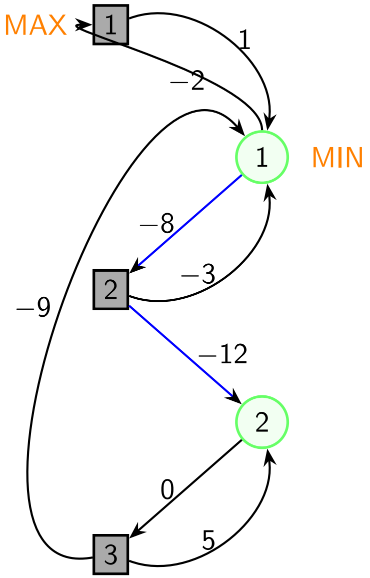

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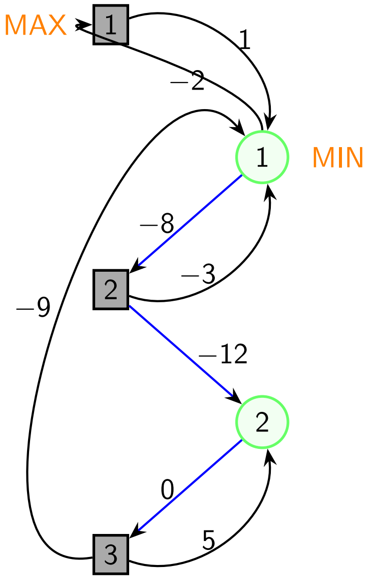

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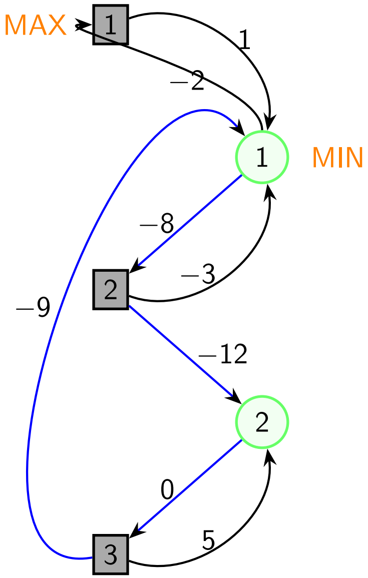

Value vector

After ℓ turns, under a strategy σ (resp. τ) of Min (resp. Max), if the sequence of actions is $i, k_1, i_1, \cdots, k_\ell, i_\ell$, the total payment:

$$R_i^\ell(\sigma,\tau) = -V_{ik_1} + V_{i_1 k_1} - V_{i_1 k_2} - \cdots + V_{i_\ell k_\ell}$$

The *value* v_i^ℓ of the game at horizon ℓ starting from i:

$$v_i^\ell := \min_\sigma \max_\tau R_i^\ell(\sigma,\tau) \ .$$

Shapley operator

Shapley operator $T : (\mathbb{R}_{\max})^n \to (\mathbb{R}_{\max})^n$,

$$T_i(x) = \min_{k\in[p], V_{ik}\neq -\infty} \Big[-V_{ik} + \max_{j\in[n], j\neq i} (V_{jk} + x_j) \Big], \quad i \in [n] \ ,$$

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- Our case: $\chi(T) \leqslant 0$.

Equivalence between tropical linear regression and MPG

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Theorem (Akian–Gaubert–Q.–Saadi)

$$\min_{a \in \mathbb{P}(\mathbb{R}_{\max})^n} \mathrm{d}_H(V, H_a) = -\rho(T) = \text{in-rad}(V)$$

Moreover,

- *if $T(a) \geqslant \rho(T) + a$, then H_a is optimal*
- *if $T(b) \leqslant \rho(T) + b$, then $B(-b, -\rho(T))$ is optimal*

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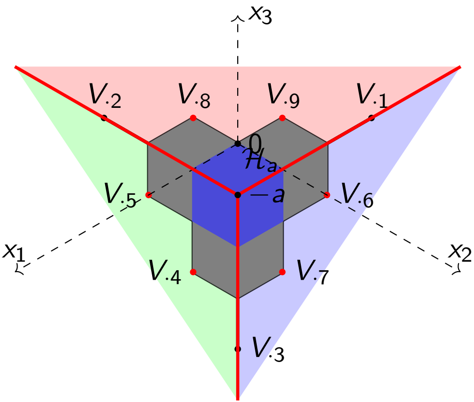
Corollary

The tropical linear regression problem is polynomial-time equivalent to the problem of solving a mean payoff game.

Geometric illustration



$$V = \begin{pmatrix} -3 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & -1 \\ 0 & -3 & 0 & 0 & -1 & 1 & 1 & -1 & 0 \\ -1 & -1 & -4 & -2 & -1 & -1 & -2 & 0 & 0 \end{pmatrix}$$

Optimality certificates

Theorem (Akian–Gaubert–Q.–Saadi)

For $a \in \mathbb{R}^n$, the following assertions are equivalent:

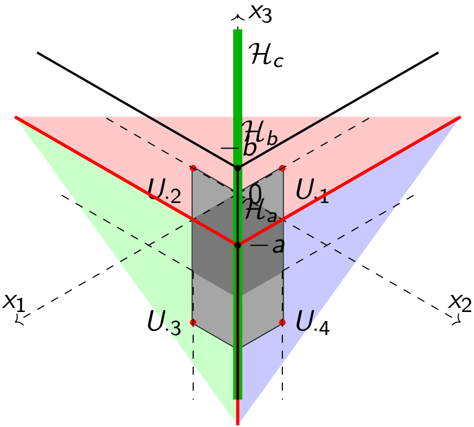
- $T(a) = \rho(T) + a$;
- *The hyperplane $\mathcal{H}_a$ admits a "witness" point in each sector, i.e.,*

$$\forall i \in [n], \exists k \in [p],\ v^{(k)} \in S_i(a) \text{ and } \operatorname{dist}_H(v^{(k)}, \mathcal{H}_a) = \operatorname{dist}_H(\mathcal{V}, \mathcal{H}_a).$$

Moreover, if these assertions hold, then

- $\rho(T) = -\operatorname{dist}_H(\mathcal{V}, \mathcal{H}_a)$
- *$\mathcal{H}_a$ is an optimal solution of the tropical linear regression problem*
- *$B(-a, \operatorname{dist}_H(\mathcal{V}, \mathcal{H}_a))$ is a Hilbert ball of maximal radius included in $\operatorname{Sp}(\mathcal{V})$.*

Nonunique optimal solutions



Col(U) with multiple optimal hyperplanes and multiple optimal inner balls, but a unique optimal hyperplane with witness points in each sector.

Revisit to equilibrium in ITT

Equilibrium:

$$\min_{i\in[q]} p_{ij} f_i^{-1} \text{ is achieved twice at least}$$

By letting $V_{ij} = -\log(p_{ij})$ and $a_i = \log(f_i)$, the equilibrium is:

$$\max_{i\in[n]}(V_{ij} + a_i) \text{ is achieved at least twice,}$$

namely,

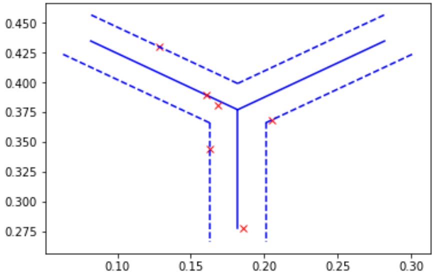
$$\forall j \in [q], \quad V_{\cdot j} \in H_a$$

In practice, we solve the **tropical linear regression problem:**

$$\min_{b\in\mathbb{P}(\mathbb{R}_{\max})^n} \mathrm{d}_H(V, H_b).$$

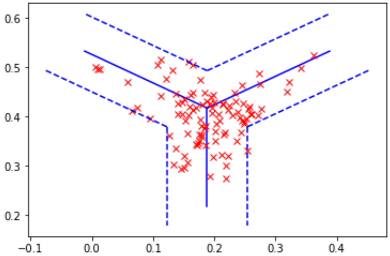
Example

	ind. houses	social housing	school	road	stadium	bridge	f	f^{reg}
Firm 1	1.02	3.21	<u>8.72</u>	26.2	69.8	<u>123</u>	1	1
Firm 2	0.81	2.65	7.49	20.3	<u>53.8</u>	106	0.8	0.81
Firm 3	<u>0.6</u>	<u>1.86</u>	5.5	<u>14.7</u>	41.8	76	0.6	0.605



Algorithm

- Goal: $T(v) = \rho(T) + v$
- Algorithm: projective Krasnoselkii-Mann value iteration algorithm
  - Given $\epsilon > 0$, start with $v^0 = (0, \cdots, 0)^\top$.
  - If $\| T(v^k) - v^k \| \geqslant \epsilon$, let
    $\tilde{v}^{k+1} = T(v^k) - (\max_{i \in [n]} T(v^k)_i)e$,
    $v^{k+1} = (1 - \beta)v^k + \beta \tilde{v}^{k+1}$,
    where $e = (1, \cdots, 1)^\top \in \mathbb{R}^n$ and $\beta \in (0, 1)$ fixed.



ϵ-approximation

Theorem (Akian–Gaubert–Q.–Saadi)

Suppose that $V \in \mathbb{R}^{n\times p}$ *is finite, and let*

$$W := \max_{k\in[p]} \|V_{\cdot,k}\|_H \ .$$

Then, an ϵ*-approximation of* in-rad(V)*, and vectors* $v, z \in \mathbb{R}^n$ *satisfying*

$$B_H(v, \text{in-rad}(\mathcal{V}) - \epsilon) \subseteq \text{Span}(V)$$

and

$$\mathrm{d}_H(\text{Span}(V), H_z) \leqslant \text{in-rad}(\mathcal{V}) + \epsilon$$

can be obtained in $O(npW/\epsilon)$ *arithmetic operations.*

Overview

Setup

- For $A \in \mathbb{R}^{n \times p}$, fix a norm $\|\cdot\|$ for the column space. Define

$$\|A\|_\phi = \phi(\|A_{\cdot 1}\|, \cdots, \|A_{\cdot p}\|),$$

where $\phi : \mathbb{R}^p \to \mathbb{R}$ under some condition.
- No SVD for tropical matrices.
- (Shitov) When $r \geqslant 7$, no polynomial time algorithm to decide if A has factor rank at most r.
- Can be computed by linear programming.

Outer radius

For a subset $\mathcal{V}$ of $(\mathbb{R}_{\max})^n$, the *outer radius* of $\mathcal{V}$ is:

$$\text{out-rad}(\mathcal{V}) := \inf\{r \geqslant 0 \mid \exists b \in \mathbb{R}^n, B(b, r) \supseteq \text{Span}(\mathcal{V})\}.$$

Theorem (Akian–Gaubert–Q.–Saadi)

For any matrix $V \in \mathbb{R}^{n \times p}$, we have

$$\min_{A \in \mathbb{R}^{n \times p}, \text{rank } A = 1} \|V - A\|_\infty = \frac{1}{2} \text{out-rad}(\text{Span } V).$$

Question: how to compute out-rad(Span V)?

Reduction to eigenvalue problem

Given $A \in (\mathbb{R}_{\max})^{n\times n}$, define

$$A^* = \mathsf{I} \oplus A \oplus A^2 \oplus \cdots$$

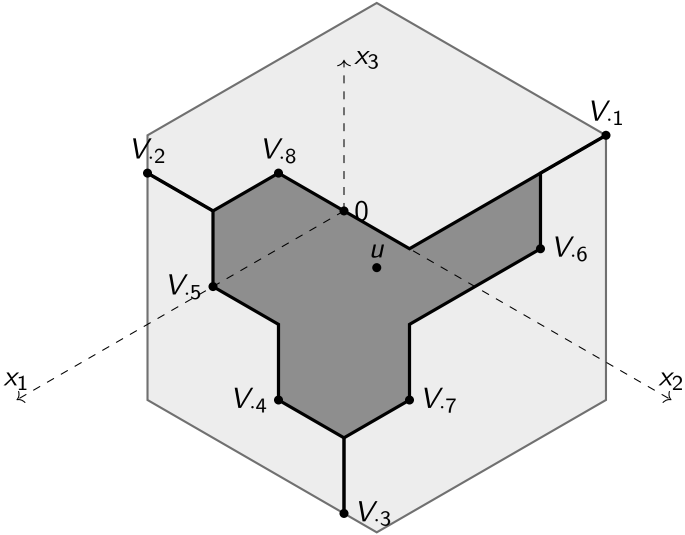
For $V \in \mathbb{R}^{n\times p}$, let $H = V \odot (-V^\top) \in \mathbb{R}^{n\times n}$, where

$$H_{ik} = \max_{j\in[p]}(V_{ij} - V_{kj}), \quad i,k \in [n].$$

Theorem (Akian–Gaubert–Q.–Saadi)

H has a unique eigenvalue, which equals out-rad(Span V)*. Moreover, the set of centers of all Hilbert outer balls of* Span V *is the column space of* $(-\lambda + H)^*$.

Example



Kernel approximations

X: compact metric, Y: nonempty, $\mathcal{C}(X)$: the space of continuous functions on X, $\mathcal{B}(Y)$: the space of bounded functions on Y.

Theorem (Akian–Gaubert–Q.–Saadi)

If $V : X \times Y \to \mathbb{R}$ bounded and $\{V(\cdot, y)\}_{y \in Y}$ equicontinuous, then

$$\inf_{f \in \mathcal{C}(X),\, g \in \mathcal{B}(Y)} \sup_{x \in X, y \in Y} |V(x,y) - f(x) - g(y)|$$

achives an optimal solution, which is equal to one half of the tropical eigenvalue of H, where

$$H(x,z) = \sup_{y \in Y} (V(x,y) - V(z,y)) \ ,$$

and f is a tropical eigenvector of H.

Rank-2 approximation

Corollary (Akian–Gaubert–Q.–Saadi)

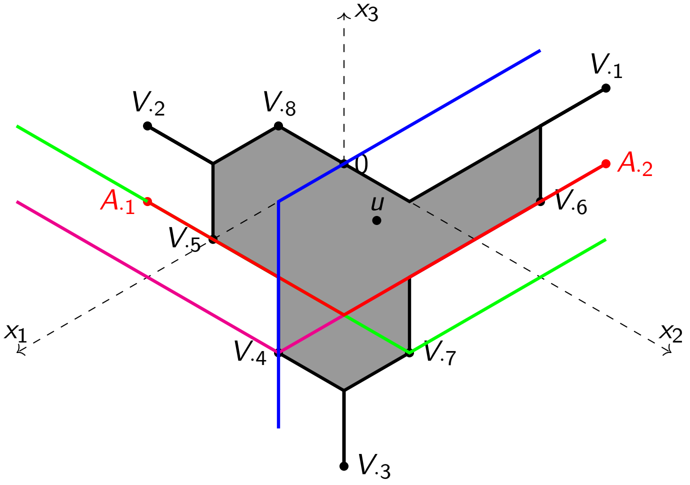
A best rank-two approximation of the matrix $V \in \mathbb{R}^{3\times p}$ *can be obtained by tropical linear regression:*

$$\min_{A\in(\mathbb{R}_{\max})^{3\times 2}, B\in(\mathbb{R}_{\max})^{2\times p}} d(V, AB) = \min_{A\in(\mathbb{R}_{\max})^{3\times 2}} \max_{k\in[p]} \mathrm{d}_H(V_{\cdot k}, \mathrm{Col}(A)) = \max_{k\in[p]} \mathrm{d}_H\left(V_{\cdot k}, \mathcal{H}_{a^*}\right) \ ,$$

and needs $O(p)$ *arithmetic operations.*

Provides us a way to compute a best rank-2 approximation for $V \in \mathbb{R}^{n\times p}$.

Example



x_3
$V_{\cdot 1}$
$V_{\cdot 2}$
$V_{\cdot 8}$
0
$A_{\cdot 2}$
$A_{\cdot 1}$
u
$V_{\cdot 6}$
$V_{\cdot 5}$
x_1
x_2
$V_{\cdot 4}$
$V_{\cdot 7}$
$V_{\cdot 3}$

Conclusion

- Tropical linear regression is equivalent to finding inner radius, which is polynomial-time equivalent to mean payoff game.
- Finding a best tropical rank-one approximation is equivalent to finding outer radius, which is equivalent to eigenvalue problem.
- A rank-2 approximation can be obtained by tropical linear regression.

Thank you for your attention!